department. The average grade for the fifty papers by the eight men was 61.4 per cent.; six of the eight instructors had average grades within 2 per cent. of this, the mean deviation for all eight being 2.1 per cent. A detailed study of the grades indicates, insofar as these grades are typical of mathematics grades in general, that if a number of instructors grade a number of papers: (*a*) The mean deviation on grades below 60 per cent. is about twice as great as on grades above 80 per cent. (*b*) The mean deviation of grades on definitions is nearly twice as great as of grades on formal algebraic work. (*c*) For good or excellent papers the mean deviation of grades on a paper containing answers to five questions is about 3 per cent., and the probability is 5 to 1 that for such a paper a random grade lies within 6 per cent. of the average. If the paper contains answers to twenty-five instead of five questions (perhaps a semester grade), this 6 per cent. is reduced to 4 per cent. (*d*) Instructors have a tendency to grade habitually high (or low), and a personal equation can be found which reduces their grades to normalcy. While the experiment may not be conclusive concerning the consistency of a group of instructors, it is pretty conclusive in testing the consistency of each with himself.

8. Dr. Kinney developed the following points: Instruction in secondary mathematics in the United States is characterized by an extreme formalism. This condition probably arises from the fact that the college mathematics of earlier times has been shoved down into the high school. There has been some improvement, as viewed from the pedagogical standpoint, during the past hundred years in the presentation of the subject matter. In geometry originals have been introduced and in algebra the so-called verbal problems. However, many of the originals have little value and many of the verbal problems do not raise legitimate mathematical questions. The writer believes that a great improvement can be made in secondary mathematical instruction by following the recommendations of the National Committee in regard to college entrance requirements.

10. Professor Ginnings discussed the effects of the present crusade against high school mathematics and the lowering of college entrance requirements in this subject. Under present conditions in Illinois, freshmen students offering two semesters of high school algebra should take a five-hour course, including algebra, the first semester and a two- or three-hour course in trigonometry the second semester. If three semesters of high school algebra is offered, then algebra and trigonometry should be offered the first semester and analytic geometry the second. Mathematics organizations should endeavor (*a*) to settle and get into high schools the best mathematics as a preparation both for college and for life (the two aims are not inconsistent); (*b*) to get a high school certificating law making it necessary for teachers to have special and ample preparation to teach the subjects they are actually allowed to teach. Future mathematical history will probably refer to the period immediately ahead of us as the "Renaissance period of the basal forms of mathematics in America."

E. B. LYTLE, *Secretary-Treasurer.*

THE MAY MEETING OF THE MINNESOTA SECTION.

The regular meeting of the Minnesota Section was held at Macalester College, St. Paul, on Saturday, May 27th. There were thirty in attendance, including the following fifteen members of the association:

W. O. Beal, W. E. Brooke, W. H. Bussey, H. H. Dalaker, Gladys E. C. Gibbens, W. L. Hart, C. A. Herrick, Dunham Jackson, R. A. Johnson, W. H. Kirchner, A. Reuterdahl, F. J. Taylor, Ella A. M. Thorp, A. L. Underhill, H. B. Wilcox.

Dinner was served at 6:30 P.M. at the Women's Dormitory of Macalester College. At the business meeting, which came after the dinner, the following officers were elected: Chairman, Professor V. H. WELLS, Carleton College; Secretary-Treasurer, Professor R. W. BRINK, University of Minnesota; Members of the executive committee, Professor W. H. KIRCHNER, University of Minnesota; L. E. LUNN, Superintendent of Schools, Heron Lake, Minnesota. At the afternoon session the following papers were read and discussed:

(1) "Single parameter mixed groups of polar fields" by Professor V. H. WELLS;

(2) "A short method of division" by Mr. SOLOMON GUTTMAN, Minneapolis (by invitation);

(3) "Logarithmic tables of the 17th and 18th centuries" by Mr. W. D. MORGAN, St. Paul (by invitation);

(4) "An analytic geometry treatment of the nature of conics generated by projective ranges and pencils" by Miss ELIZABETH CARLSON, University of Minnesota (by invitation);

(5) "Nomograms" by Professor R. W. BRINK;

(6) "The Finite and Infinite, Space and Time" by Professor ARVID REUTERDAHL.

Abstracts of the papers are given below, the numbers corresponding to the numbers in the list of titles:

1. Professor Wells gave a development of three single parameter systems of polar fields which lie between the pencil and the range of polar fields. The polars of a point envelop a conic or a fourth class curve and the poles of a line lie on a fourth order curve or on a conic.

2. Mr. Guttman presented a new arrangement of arithmetical long division in which the usual subtractions are turned into additions. This is done by making use of the arithmetical complement, defined as $(10^n - D)$, where D is the divisor and n the number of digits in the divisor. The chief advantage of the method consists in a check for accuracy which depends on the identity $A + Q(10^n - D) = Q \cdot 10^n + R$, A being the dividend, Q the quotient and R the remainder. *Example:* The compact form of the division of 48375 by 57, by this method, would be as follows:

57	48375
43	344
	172
	344
	84839

The quotient is 848 and the remainder is 39.

3. Mr. Morgan's paper concerned the history and construction of five of the earlier logarithmic tables, namely, Henry Briggs, *Arithmetica Logarithmica*, London, 1624; Adrian Vlacq, *Arithmetica Logarithmica*, Goudae, 1628; Briggs-Gellibrand, *Trigonometria Britannica*, Goudae, 1633; Dodson, *Anti-Logarithmic Canon*, London, 1742, and Vega, *Thesaurus Logarithmorum*, Leipzig, 1794. Copies of each of these works which are now all entirely out of print and very scarce were exhibited at the meeting. The *Arithmetica Logarithmica* of Briggs and the *Trigonometria Britannica* are the only fourteen place tables ever published. The *Anti-Logarithmic Canon* is an eleven place table of anti-logarithms, the most extensive ever constructed. The *Arithmetica Logarithmica* of Vlacq and the *Thesaurus* of Vega are both ten place tables. The latter gives also the Wolframmi table of Naperian logarithms to forty-eight decimal places.

4. The nature of the conic generated by projective ranges and pencils, as determined by the projective relation involved in its generation, has been obtained by Steiner by the use of synthetic methods. In this paper Miss Carlson obtained the same and other results by the use of analytic geometry methods.

5. In his paper Professor Brink explained the purpose of nomograms and the method of using them. He exhibited nomograms for the solution of right triangles and other problems, and indicated their usefulness in teaching college mathematics.

6. Professor Reuterdahl discussed the significance of Space and Time for the Finite and Infinite of mathematics. Space and Time may be regarded as the cosmic background of all knowledge. The finite and infinite are inseparable by-products of Space-Time. As a dual cosmic principle, Space-Time is a dynamic totality. Time is the space-separator and space is the time-binder. The onward urge of time conveys the impression of incompleteness and thus time constitutes the dynamic element of the dual principle Space-Time. Time is like a shearing blade which dissects space. Time can shear space only in three distinct ways and therefore physical space is three dimensional. Spatial manifolds of higher dimensionality are merely conceptual spatial extensions. Space-Time constitutes the only true continuum in the cosmos. The study of the finite and the infinite, consequently, becomes an investigation of the nature of this dynamic continuum. The paradox of a process-infinite composed of continuously associated finite phases finds its solution in the dynamic continuum of Space-Time.

GLADYS GIBBENS, *Acting Secretary.*

A SIMPLE FORM OF DUHAMEL'S THEOREM AND SOME NEW APPLICATIONS.[1]

By H. J. ETTLINGER, University of Texas.

1. Introduction. It is an open question whether Duhamel's[2] theorem should be given to a beginning class in the calculus. That one can avoid the use of this

[1] Read before the Mathematical Association of America, September, 1921.

[2] Duhamel, *Eléments de calcul infinitésimal*, Paris, 1856, p. 35.

theorem in dealing with *some* problems involving definite integrals such as length, volume, pressure, etc., must be recognized.[1] On the other hand, Duhamels' theorem is such a powerful tool for use in evaluating the limit of a sum by means of a definite integral and provides such a simple test as to when one infinitesimal may be replaced by another of the same order in this type of problem, that it is highly desirable to have it among the elementary theorems of the worker in mathematics. The theorem, furthermore, has important applications to the transformation of a double integral and the solution of integral equations.

W. F. Osgood [2] was the first to call attention to a rigorous formulation of the theorem. His proof makes use of uniform convergence to a limit and does not readily adapt itself to elementary treatment. The proof itself, based on Abel's lemma, is rather direct and simple,[3] though it is generally accepted that a proof involving uniform convergence should be taboo in a first course in calculus. However direct the proof may be, the *application* of the theorem is too difficult for a beginner and is not attempted in this form by Osgood in his text-book.[4]

R. L. Moore[5] has given a very general form of Duhamel's theorem whose importance, because of its generality, has been entirely overlooked. Some of the later writers simply mention Moore's form in passing. G. A. Bliss[6] presented a theorem equivalent to Duhamel's but involving the concept of uniform continuity. Recently E. V. Huntington[7] and H. B. Fine[8] presented simplified forms of Bliss's substitute. Finally G. James[9] has added a substitute different in form from the others.

It is the object of this paper to present Duhamel's theorem by means of a geometric lemma, which has the following advantages: (a) it is so simple in form that the beginning student in calculus will, from intuition, accept it as correct, subject to the statement that it will be proved in a later course; (b) it avoids the question of double limits and its concomitant concept, uniform convergence;[10]

[1] Cf. a suggestion by one who has "suffered" from the effects of Duhamel's theorem, Benjamin Graham, "Some calculus suggestions by a student," in this MONTHLY, *1917*, pp. 265–271.

[2] "The integral as the limit of a sum, and a theorem of Duhamel's," *Annals of Mathematics*, second series, vol. 4, 1903, pp. 161–178.

[3] Cf. W. F. Osgood, *A First Course in the Differential and Integral Calculus*, New York, 1907, pp. 164–165. The uniformity condition in the statement of the theorem is here omitted.

[4] *Ibid.*, pp. 166, 167, 168, 171, 173, 178, 182. In the applications no test of uniform convergence is made.

[5] "On Duhamel's theorem," *Annals of Mathematics*, second series, vol. 13, 1912, pp. 161–166. This paper, though mentioned by Bliss, Huntington and James, does not seem to have attracted the attention it deserves.

[6] "A substitute for Duhamel's theorem," *Annals of Mathematics*, second series, vol. 16, 1914, pp. 45–49.

[7] "On setting up a definite integral without the use of Duhamel's theorem," this MONTHLY, *1917*, pp. 271–275.

[8] "Note on a substitute for Duhamel's theorem," *Annals of Mathematics*, second series, vol. 19, 1918, pp. 172–173.

[9] "A substitute for Duhamel's theorem," *Tôhoku Mathematical Journal*, vol. 17, 1920, pp. 7–9.

[10] The secretary's report of the Wellesley meeting records an observation having been made indicating doubt as to the truth of this statement (see *1921*, 359–360). Responsibility for the origin of this observation could not be fixed. That there is no basis for this doubt is evident

(c) it is more general than the other forms;[1] (d) it is easily applied. In 2 the various forms of Duhamel's theorem are set forth for comparison. In 3 the geometric lemma is stated and Duhamel's theorem derived from it. In 4 an application is made to prove the existence of a definite integral for a continuous function and an example worked out for the usual class of problems in definite integrals. In 5 indications are given of applications of a more general form of the theorem to the transformation of a double integral and the solution of integral equations.

2. Various Forms of the Theorem. Osgood's[2] form of Duhamel's theorem is the following:

"*Let*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad (A)$$

be a sum of infinitesimals and let α_i *differ uniformly by an infinitesimal of higher order than* Δx_i *from the summand* $f(x_i)\Delta x_i$ *of the definite integral*

$$\int_a^b f(x)dx \qquad (B)$$

of the function $f(x)$, *this function being continuous throughout the interval* $a \leqq x \leqq b$. *Then the sum* (A) *approaches a limit when* $n = \infty$, *and the value of this limit is the definite integral* (B):

$$\lim_{n=\infty} \sum_{i=1}^{n} \alpha_i = \int_a^b f(x)dx."$$

The application of this theorem to any problem would call for a proof that

$$\lim_{n=\infty} \left[\frac{\alpha_k}{\Delta x} - f(x_k) \right] = 0$$

uniformly with respect to k. This is not to be done with ease in particular examples.

The following is Moore's[3] form:

"HYPOTHESIS: (*a*) E *is a limited point-set in a space of n-dimensions.* $E_{1n}, E_{2n}, \cdots, E_{nn}$ *are (for each value of the positive integer* n*) non-overlapping sub-sets of* E *of interior measures* $e_{1n}, e_{2n}, \cdots, e_{nn}$, *respectively.* r_{in}, r_{in}' $(i = 1, \cdots, n)$ *are numbers such that the set* $\{|r_{in}' - r_{in}|\}$ *is a bounded set, i.e., there exists a number* c *such that for all values of* n *and* i $(i \leqq n)$, $|r_{in}' - r_{in}| \leqq c$. (*b*) $\operatorname{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$ *exists.* (*c*) E_0 *is a subset of* E *of measure* 0. (*d*) *If* P *is a point of* E *not belonging to* E_0, *then*

$$\operatorname{Lim}_{n=\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0.$$

upon reading the proofs of the lemma. Also compare Osgood's theorem on non-uniformly convergent series, p. 244, footnote, and Moore's theorem, p. 241, referred to in this paper.

[1] Moore's form, of which the lemma is a corollary, is, of course, to be excepted.

[2] See *Annals* paper referred to above, p. 173.

[3] *L.c.*, pp. 162–163.

CONCLUSION: $\operatorname{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}'e_{in}$ *exists and equals*

$$\operatorname{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}.$$"

That this theorem is more general than Osgood's form has been shown by an independence example by Moore.[1] Unfortunately the extreme generality and nomenclature of Moore's theorem has caused its importance to be overlooked.

The substitute theorem of Bliss[2] is the following:

"*Let us consider a function of the form* $f(p, p', p'')$ *where* p *is a symbol for a set of values* (x, y, z) *and* p' *and* p'' *for analogous sets. The points* p, p', p'' *are to range over a closed measurable region* V *in* xyz*-space in which* f *is continuous and hence uniformly continuous.*

"*If the region* V *is divided into measurable sub-regions with maximum diameters less than* δ *and with volumes denoted by* ΔV_k $(k = 1, 2, \cdots, n)$, *and if in each region three points* p_k, p_k', p_k'' *are chosen, then*

$$\operatorname{Lim}_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k', p_k'')\Delta V_k = \operatorname{Lim}_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k, p_k)\Delta V_k = \int_V f(p, p, p)dV."$$

The proof depends on the fact that f is uniformly continuous, a concept which Bliss frankly admits has little meaning to the average sophomore, but which he hopes will "leave a framework of proof in the mind of the student."

Huntington[3] simplifies Bliss's theorem in the following form:

"*Suppose that a required quantity* P *is associated with a real interval,* $x = a$ *to* $x = b$, *in such a way that we are led to divide the interval into* n *small parts or 'elements,'* Δx, *and to regard* P *as the sum of* n *separate contributions, one from each element. Suppose also that a set of one or more functions,* $F(x), f(x), \cdots$, *can be found, such that, no matter what value of* x *is considered, and no matter how small* Δx *may be, the contribution from a typical element,* $x = x$ *to* $x = x + \Delta x$, *can be expressed 'approximately' (see note 1) in the form*

$$[F(x)f(x) \cdots]\Delta x.$$

Then the required quantity P *will be correctly given by the value of the definite integral*

$$P = \int_a^b [F(x)f(x) \cdots]dx,$$

whenever the functions $F(x), f(x), \cdots$ *are continuous from* $x = a$ *to* $x = b$.

"*Note 1.* The word 'approximately' is here used in a technical sense, meaning that the exact value of the contribution in question lies between $[\overline{F} \cdot \overline{f} \cdots]\Delta x$ and $[\underline{F} \cdot \underline{f} \cdots]\Delta x$, where $\overline{F}, \overline{f}, \cdots$ are the smallest and $\underline{F}, \underline{f}, \cdots$ the largest values of $F(x), f(x), \cdots$ in the element."

Fine[4] also has given a simplification of Bliss's substitute. His statement is as follows:

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[1] *L.c.*, p. 166.

[2] *L.c.*, p. 46.

[3] *L.c.*, pp. 273 and 274. The proof as given in the footnote 1 on p. 273 is different from Bliss's.

[4] *L.c.*, p. 172.

"*Let* $f_1(x), f_2(x), \cdots, f_p(x)$ *denote any set of functions of* x, *finite in number, which are continuous in the interval* (ab) *and let*

$$F(x) = f_1(x)f_2(x) \cdots f_p(x).$$

"*Suppose the interval* (ab) *to be divided and redivided into parts in any manner such that as the process is indefinitely continued the greatest of the parts will approach* 0 *as a limit, and at any stage in the process let* $h_1, h_2, \cdots, h_n$ *represent the parts in length and position; also let* $\xi_i', \xi_i'', \cdots, \xi_i^{(p)}$ *and* ξ_i *denote any numbers in the part* h_i $(i = 1, 2, \cdots, n)$. *Then*

$$\lim_{n \doteq \infty} \sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'') \cdots f_p(\xi_i^{(p)})h_i = \lim_{n \doteq \infty} \sum_{i=1}^{n} F(\xi_i)h_i = \int_a^b F(x)dx."$$

The proof[1] of this theorem, like the proof of Bliss's, is based on uniform continuity.

James[2] gives the following substitute:

"*Divide the interval, a to b, into n positive subintervals,* Δx_i, $i = 1, 2, 3, \cdots, n$, *each of which converges to zero as n increases. If*

$$\lim_{n \to \infty} \sum_{1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx,$$

then

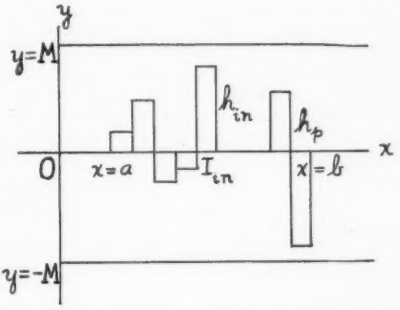
$$\lim_{n \to \infty} \sum_{1}^{n} [f(x_i)\Delta x_i + \phi_i(n)\Delta x_i]$$

exists and is equal to

$$\int_a^b f(x)dx,$$

provided there exists a sequence of constants $c_1, c_2, c_3, \cdots$ *(independent of* n*),*[3] *with limit zero, such that* $|\phi_i(n)| \leqq c_i$, $i = 1, 2, \cdots n$."

This form though more general than Osgood's is less general than Moore's, and the simplicity of the proof and ease of application are debatable.



3. A Geometric Lemma and a Simple Form of Duhamel's Theorem. The following lemma is stated in geometric form because it is believed that it will appeal to the student's intuition and will be one which he will readily accept, subject to the remark that the proof must be deferred until a later time.

LEMMA: — *Let the interval* $I : a \leqq x \leqq b$ *be divided into n equal closed subdivisions,* I_{in}, *of length,* $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. *Upon* I_{in} *as base*

[1] *L.c.*, p. 173.

[2] *L.c.*, pp. 7–8.

[3] Parenthesis inserted by the present writer.

draw a rectangle of area, R_{in}, and height, h_{in}, such that

$$|h_{in}| \leqq M \quad (i = 1, 2, \cdots n)$$

for all values of n, where M is a constant independent of i and n. If $\overline{P}$ is any fixed point of I, there is for each value of n at least one rectangle whose base contains $\overline{P}$. If for each fixed $\overline{P}$ the altitude of this rectangle, $h_{\overline{P}}$, approaches 0 as a limit as $n \to \infty$, $\lim_{n\to\infty} h_{\overline{P}} = 0$, then $\lim_{n\to\infty} \sum_{i=1}^{n} R_{in} = 0$.

The proof[1] of this lemma is omitted, because it has no immediate relation to the purpose of this paper, but references to proofs are given in the footnote. The acceptance of the above lemma as true may be compared to the acceptance of Rolle's theorem as true from geometric intuition, or the acceptance of the following theorem: *If $f(x)$ is continuous in $a \leqq x \leqq b$ and $f(a) \neq f(b)$, then the equation $f(x) = N$, where N is between $f(a)$ and $f(b)$, has at least one root in $a < x < b$.* In fact Huntington[2] explicitly assumes this latter theorem without proof, and Graham,[3] who also uses it, refers to Goursat's *Cours d'Analyse Mathématique* for a proof.

On the basis of this geometric lemma we can readily establish

DUHAMEL'S THEOREM:—*Let the interval $I : a \leqq x \leqq b$ be divided into n equal subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base draw two rectangles, R_{in}' and R_{in}'', of height h_{in}' and h_{in}'', respectively, such that*

$$|h_{in}' - h_{in}''| \leqq M \quad (i = 1, 2, \cdots n)$$

for all values of n, where M is a constant, independent of i and n. If $\overline{P}$ is any fixed point of I, there is for each value of n at least one rectangle, R_{in}', whose base contains $\overline{P}$ and a corresponding rectangle, R_{in}'', with the same base. If, for each fixed $\overline{P}$, the difference of the altitudes of these rectangles, $h_{\overline{P}}'$ and $h_{\overline{P}}''$, approaches zero as a limit,

$$\lim_{n\to\infty} (h_{\overline{P}}' - h_{\overline{P}}'') = 0,$$

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[1] A proof may be given similar to Moore's (*l.c.*, p. 163), based on the application of a theorem due to W. H. Young (*Proceedings of the London Mathematical Society*, series 2, vol. 2, 1905, p. 25). Still another proof is that of E. Landau (*Mathematische Zeitschrift*, vol. 2, 1918, pp. 350–351). Professor Osgood, in a recent letter, pointed out that the lemma is virtually contained in a theorem which he proved in the *Amer. Jour. of Math.*, vol. 19, 1897, p. 188. He showed that, *if $s_n(x)$ be continuous in the interval $a \leqq x \leqq b$ for all values of n, and if $s_n(x)$ approach a continuous limit; if, furthermore, $s_n(x)$, regarded as a function of the two independent variables x and n, remain finite, then*

$$\lim_{n=\infty} \int_a^b s_n(x)dx = \int_a^b \lim_{n=\infty} s_n(x)dx.$$

From this theorem the lemma follows at once. Let $s_n(x)$ be defined as follows: On I_{in} as base erect an isosceles triangle whose altitude is $2h_{in}$. The equal legs of these triangles shall form the graph of $s_n(x)$. And now the area under $s_n(x)$ is precisely $\sum_{i=1}^{n} R_{in}$. But $\lim_{n=\infty} s_n(x) = 0$. Hence $\lim_{n=\infty} \sum_{i=1}^{n} R_{in} = 0$.

Since this paper was written the author has given an elementary proof based on the usual fundamental limit theorems of the beginning of the calculus (presented to the American Mathematical Society, September 7, 1922).

[2] *L.c.*, p. 273, footnote 1.

[3] *L.c.*, p. 268, Theorem 3, Lemma.

and if $\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n$ *exists, then* $\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}''\Delta x_n$ *exists and*

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n = \lim_{n\to\infty}\sum_{i=1}^{n} h_{in}''\Delta x_n.$$

Let

$$h_{in} = h_{in}' - h_{in}''.$$

Then $|h_{in}| \leqq M$ for all values of n, and $\lim_{n\to\infty} h_{\bar{P}} = 0$ for every fixed P. Hence by the lemma,

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}\Delta x_n = 0 \qquad \text{or} \qquad \lim_{n\to\infty}\sum_{i=1}^{n} (h_{in}' - h_{in}'')\Delta x_n = 0.$$

Hence

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n = \lim_{n\to\infty}\sum_{i=1}^{n} h_{in}''\Delta x_n.$$

4. Applications to Definite Integrals. We proceed to use the theorem of 3 to show that if $f(x)$ is continuous in the interval $I: a \leqq x \leqq b$, then $f(x)$ is integrable in I. Let I be subdivided into n equal subdivisions, I_{in} $(i = 1, 2, \cdots n)$, of length $\Delta x_n = (b-a)/n$. Let H be the maximum value of $f(x)$ in I; and h the minimum value of $f(x)$ in I. Let $\bar{P}$ be any fixed point of I and let $I_{\bar{P}n}$, for each value of n, be a closed subinterval I_{in} which contains $\bar{P}$. Let $H_{\bar{P}n}$ be the largest value of $f(x)$ in $I_{\bar{P}n}$ and let $h_{\bar{P}n}$ be the smallest value of the set $H_{\bar{P}1}$, $H_{\bar{P}2}, \cdots, H_{Pn}$. Let h_{in} be defined for each I_{in} as the number which for each $\bar{P}$ in I_{in} is equal to the value of $h_{\bar{P}n}$ which corresponds to $\bar{P}$. For each point $\bar{P}$, h_{in} never increases as n increases. Hence $A_n = \sum_{i=1}^{n} h_{in}\Delta x_n$ never increases as n increases. But $A_n \geqq h(b-a)$. Therefore as $n \to \infty$, A_n approaches a limit or $\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}\Delta x_n$ exists.

Let ξ_{in} be any value of x in I_{in} and $\bar{\xi}_{in}$, the value of ξ_{in} which corresponds to $I_{\bar{P}n}$. Now $\lim_{n\to\infty} h_{\bar{P}n} = f(\bar{x})$ where $\bar{x}$ is the abscissa of $\bar{P}$, and $\lim_{n\to\infty} f(\bar{\xi}_{in}) = f(\bar{x})$ since $f(x)$ is continuous in I. Also $|h_{in} - f(\xi_{in})| \leqq H - h$ for every value of $i(\leqq n)$ and n. Hence by the theorem of 3,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_{in})\Delta x_n$$

exists. This limit is independent of the choice of ξ_{in}, and of the method of subdivision.

Hence by the usual argument

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_{in})\Delta x_n = \int_a^b f(x)dx.$$

We may now restate Duhamel's theorem:

Let $f(x)$ be a continuous function of x in the interval $I: a \leqq x \leqq b$. Let I be subdivided into n equal closed subdivisions, I_{in}, of length, $\Delta x_n = (b-a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base draw a rectangle of height β_{in}. Let ξ_{in} be any value of x in I_{in}. If $\bar{P}$ is any fixed point of I whose $x = \bar{x}$, there is for each value of n at least one rectangle with height, $\beta_{\bar{P}}$, whose base $\bar{I}_{in}$ contains $\bar{P}$. If $|\beta_{in}| \leqq M$ $(i = 1, 2, \cdots n)$ for all values of n, where M is a constant, independent of i and n, and if for each fixed point $\bar{P}$ $\lim_{n\to\infty} \beta_{\bar{P}} = f(\bar{x})$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \beta_{in}\Delta x_n = \int_a^b f(x)dx.$$

If we identify $f(\xi_{in}) = h_{in}'$ and $\beta_{in} = h_{in}''$ of the theorem in 3, the derivation of the preceding theorem is immediate.

For the application of this form of Duhamel's theorem to the usual problems of the first course of the calculus, it is only necessary to refer to the examples in Osgood's[1] text-book with some modifications. In each of the examples cited the theorem of 3 is tacitly assumed by Osgood and applied. For $x = x_k$ in these instances cannot be taken as designating the right hand end of the kth subdivision since $\lim_{n\to\infty} x_k = a$ and $\lim_{n\to\infty} (\beta_k/\alpha_k) = 1$ (where $\beta_k = \beta_{kn}\Delta x_n$, $\alpha_k = f(x_{kn})\Delta x_n$ in our notation) become meaningless as far as the application of Osgood's form of the theorem is concerned. Professor Osgood has indicated in another letter that he regards x_k as fixed. This is impossible unless k varies with n and n increases without limit in some special manner. What is evidently intended is that x_k shall be regarded as fixed entirely independent of any change in n and, furthermore, that it shall represent any value of x in I, *i.e.*, x_k is the $\bar{x}$ of our theorem. It is also desirable to replace $\lim_{n\to\infty} (\beta_k/\alpha_k) = 1$ by $\lim_{n\to\infty} (\beta_k - \alpha_k) = 0$, in our notation $\lim_{n\to\infty} (h_{\bar{P}}' - h_{\bar{P}}'') = 0$, since this avoids the difficulty of $\alpha_k = 0$. With these interpretations we reproduce the example on p. 166 in Osgood's text-book.

To find the length of the curve $y = F(x)$ from $x = a$ to $x = b$. For simplicity let $F(x)$ be continuous with a continuous derivative in $I: a \leqq x \leqq b$. Subdivide I into n equal parts; erect ordinates at the points of division; and inscribe a broken line in the arc to be measured. The length of this line is

$$\sum_{i=0}^{n-1} \sqrt{\Delta x_n^2 + \Delta y_{in}^2} = \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}\,\Delta x_n.$$

Now, to make connections with the foregoing theorem, let

$$f(x) = \sqrt{1 + [F'(x)]^2}$$

and

$$\beta_{in} = \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}.$$

[1] *L.c.*, see examples on pp. 166, 167, 168, 171, 173, 178, 182.

Now $|\beta_{in}| = \sqrt{1+\left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}$ remains finite and $\lim_{n\to\infty} \beta_{\bar{P}} = \sqrt{1+[F'(\bar{x})]^2} = f(\bar{x})$. Hence the conditions of Duhamel's theorem are satisfied and

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta x_n{}^2 + \Delta y_n{}^2} = \int_a^b \sqrt{1+[F'(x)]^2}\, dx.$$

It seems to the writer that the form of Duhamel's theorem here given, on account of its simplicity and ease of application, should earn a place in our elementary calculus texts.[1]

5. Applications to the Transformation of a Double Integral and the Solution of an Integral Equation.[2] The geometric lemma and the statement of Duhamel's theorem of 3 may be generalized in several directions. It is quite obvious that it is not essential that the lengths of the subdivisions be equal, and instead of a linear interval we may have a closed space of any number of dimensions. These indications are merely special cases of Moore's general form which was reproduced in 2. We proceed to apply this form to the transformation of a double integral.[3]

HYPOTHESES: 1. *$f(x, y)$ is a function having an upper and lower bound defined for all points (x, y) of a closed, connected domain, G, contained in H, a bounded connected domain, the boundary of G having content zero;*

2. *x and y are expressed in terms of ξ and η by the relations*

$$x = f_1(\xi, \eta), \qquad y = f_2(\xi, \eta), \tag{T}$$

such that the transformation (T), as well as its inverse

$$\xi = \phi_1(x, y), \qquad \eta = \phi_2(x, y), \tag{T'}$$

is continuous and one-to-one. Let $F(\xi, \eta) \equiv f[f_1(\xi, \eta), f_2(\xi, \eta)]$ and let G' in H' represent the transform of G in H. L is a set of points of G of measure zero;

3. *$f(x, y)$ is continuous in G except at points of L;*

4. *the partial derivatives $\frac{\partial f_1}{\partial \xi}, \frac{\partial f_1}{\partial \eta}, \frac{\partial f_2}{\partial \xi}, \frac{\partial f_2}{\partial \eta}$ exist and are continuous at every point of H', except at points of the set L' corresponding to L;*

5. *the Jacobian of $f_1(\xi, \eta)$, $f_2(\xi, \eta)$ with respect to ξ and η, $J = \frac{\partial(f_1, f_2)}{\partial(\xi, \eta)}$, is different from zero in H', except at points of L'.*

[1] The ordinary "carefree" treatment of Duhamel's theorem may be exemplified in an otherwise most excellent text, G. A. Gibson, *Elementary Treatise on the Calculus*, London, 1919. In many respects this book is superior to the average calculus text-book. On p. 198 the author states that Duhamel's theorem "is not necessarily true if the infinitesimals are not all of the same sign." That this is incorrect is established by the present paper. He adds that "from its use in integration this theorem is often called the fundamental theorem of the integral calculus."

[2] Since this paper was written, the author has made other important applications (1) to the transformation of a simple integral, (2) to the differentiation or integration termwise of a non-uniformly convergent series, (3) (by a pupil) to the approximate solution of differential equations, (4) to Green's theorem in potential theory and related problems.

[3] See E. W. Hobson, *The Theory of Functions of a Real Variable*, Cambridge, 1907, pp. 445–452.

CONCLUSION:

$$\iint_G f(x, y)dxdy = \iint_{G'} F(\xi, \eta) \left| \frac{\partial(f_1, f_2)}{\partial(\xi, \eta)} \right| d\xi d\eta.$$

The outline of the proof is as follows:

$$\begin{aligned} \iint_G f(x, y)dxdy &= \lim_{n\to\infty} \sum_{i=1}^{n^2} f(x_{in}, y_{in})\Delta G_{in} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n^2} f[f_1(\xi_{in}, \eta_{in}), f_2(\xi_{in}, \eta_{in})]\Delta G_{in} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n^2} F(\xi_{in}, \eta_{in}) \frac{\Delta G_{in}}{\Delta G_{in}'} \cdot \Delta G_{in}'. \end{aligned}$$

If $\overline{P}(\bar{x}, \bar{y})$ is a fixed point of G not in L, then for each value of n there is (at least) one $\Delta\overline{G}_{in}$ containing $\overline{P}$. If $\Delta\overline{G}_{in}'$ is the transform of $\Delta\overline{G}_{in}$ by T, then it may be proved that

$$\lim_{n\to\infty} \left[\frac{\Delta\overline{G}_{in}}{\Delta\overline{G}_{in}'} - J_{|\xi=\bar{\xi}, \eta=\bar{\eta}|} \right] = 0^1 \qquad \text{and} \qquad \left| \frac{\Delta G_{in}}{\Delta G_{in}'} - J_{|\xi=\bar{\xi}, \eta=\bar{\eta}|} \right| \leqq M$$

for each value of n.

Hence by Moore's form of Duhamel's theorem the transformation is established.

Fredholm[2] solved the integral equation of the second kind,

$$u(x) = f(x) + \int_a^b K(x, \xi)u(\xi)d\xi, \tag{E}$$

by means of a system of algebraic equations in an infinite number of unknowns. We give Bôcher's[3] treatment:

Let the interval $I : a \leqq x \leqq b$ be subdivided into n equal subdivisions, I_{in} $(i = 1, 2, \cdots n)$, of length $\Delta x_n = (b - a)/n$ and let $x_{0n} = a, x_{1n}, x_{2n}, \cdots, x_{nn} = b$ designate the points of division. If we replace (E) by the equation

$$u(x) = f(x) + \sum_{j=1}^{n} K(x, x_{jn})u_n(x_{jn})\Delta x_n, \tag{E'}$$

(E') is to be satisfied for the values x_{in} $(i = 1, 2, \cdots n)$. This yields the system of n equations

$$u_n(x_{in}) = f(x_{in}) + \sum_{j=1}^{n} K(x_{in}, x_{jn})u_n(x_{jn})\Delta x_n \quad (i = 1, 2, \cdots n) \tag{S}$$

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[1] Cf. Goursat, *Cours d'Analyse Mathématique*, Paris, 1902, vol. 1, pp. 300–302.

[2] "Sur une nouvelle méthode pour la résolution du problème de Dirichlet," *Öfversigt af Kungl. Vetenskaps-Akademiens Förhandlingar*, vol. 57, 1900, p. 39. Also *Acta Mathematica*, vol. 27, 1903, pp. 365–390.

[3] *Cambridge Tracts in Mathematics and Mathematical Physics*, no. 10, *An Introduction to the Study of Integral Equations*, Cambridge, at the University Press, 1909, pp. 25–27.

in n unknown quantities, $u_n(x_{in})$, $(i = 1, 2, \cdots n)$. (S) may be written as

$$-\sum_{j=1}^{n} K(x_{in}, x_{jn})u_n(x_{jn})\Delta x_n + u_n(x_{in}) = f(x_{in}) \quad (i = 1, 2, \cdots n). \qquad (S')$$

By Cramer's formula the value of $u_n(x_{\mu n})$ is

$$u_n(x_{\mu n}) = \frac{1}{D_n}\sum_{i=1}^{n} f(x_{in})D_n(x_{\mu n}, x_{in}) \quad (\mu = 1, 2, \cdots n),$$

where $D_n(\neq 0)$ is the determinant of the system [1] and $D_n(x_{\mu n}, x_{\nu n})$ is the cofactor of the element in the νth row and the μth column. The solution of (E) is now obtained by letting n become infinite and at the same time allowing μ to vary with n in such a manner that $x_{\mu n}$ approaches a fixed value x in I. If $D = \lim_{n\to\infty} D_n$ and $D(x, \xi) = \lim_{n\to\infty} \frac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$, then it may be proved that

$$u(x) = f(x) + \frac{1}{D}\int_a^b f(\xi)D(x, \xi)d\xi.$$

This is the solution of (E).

The application of Duhamel's theorem in this proof is that of determining the limiting form of $\frac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$. The hypothesis [2] concerning $K(x, \xi)$ is that it shall be bounded in $S : a \leqq x \leqq b$, $a \leqq \xi \leqq b$ and continuous except for a set of points, L, of measure zero, and $K(x, x)$ shall be integrable in $I : a \leqq x \leqq b$. The typical term [3] of $\frac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$, except for the omission of the numerical coefficient, is

$$\underbrace{\sum_{i, j, \cdots = 1}^{n}}_{m-1} \begin{vmatrix} K(x_{\mu n}, x_{\nu n}) & K(x_{\mu n}, x_{in}) & K(x_{\mu n}, x_{jn}) \cdots & m \text{ columns} \\ K(x_{in}, x_{\nu n}) & K(x_{in}, x_{in}) & K(x_{in}, x_{jn}) & \cdot \\ K(x_{jn}, x_{\nu n}) & K(x_{jn}, x_{in}) & K(x_{jn}, x_{jn}) & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m \text{ rows} & \cdot & \cdot & \cdot \end{vmatrix} \overline{\Delta x_n}^{m-1}. \qquad (A)$$

If (x, ξ) is a fixed point in S not in L such that as n increases $x_{\mu n}$ approaches x and $x_{\nu n}$ approaches ξ, then the limit of (A) would appear to be

$$\underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{m-1} \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & K(x, \xi_2) \cdots & m \text{ columns} \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & K(\xi_1, \xi_2) & \cdot \\ K(\xi_2, \xi) & K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m \text{ rows} & \cdot & \cdot & \cdot \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_{m-1}. \qquad (B)$$

[1] Bôcher, *l.c.*, p. 26.
[2] *L.c.*, p. 36, theorem 3.
[3] *L.c.*, p. 27.

That this is the limit can be proved by identifying (A) above with $\Sigma r_{in}'e_{in}$ and (B) with $\lim_{n\to\infty} \Sigma r_{in}e_{in}$ of the theorem of p. 241.

Now $K(x, \xi)$ is bounded; hence

$$|r_{in}' - r_{in}| \leqq C$$

and $\lim_{n\to\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0$, if P is not a point of L, since $K(x, \xi)$ is continuous except for points of L. Hence Duhamel's theorem establishes the limiting form of (A) to be (B).

NOTE ON APPLICATION OF DIOPHANTINE ANALYSIS TO GEOMETRY.

By HORACE L. OLSON, University of Michigan.

It frequently becomes desirable, in teaching analytic geometry of three dimensions, to know a set of three mutually perpendicular lines each of which has all its direction cosines rational. It is well known that if two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively, the direction cosines of their common perpendicular are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, and $(l_1m_2 - l_2m_1)$, and that if the two first-mentioned lines are perpendicular the factor of proportionality is unity. Hence, if two of our three mutually perpendicular lines have rational direction cosines, the third will also have this property. Furthermore, it will presently appear that one of these lines can be taken to be any line having rational direction cosines.

We therefore have first to find rational solutions of the equation

$$l_1^2 + m_1^2 + n_1^2 = 1. \tag{1}$$

It is well known, and can easily be verified, that all such solutions are given by the formulæ

$$l_1 = \frac{p^2 + q^2 - r^2}{p^2 + q^2 + r^2}, \qquad m_1 = \frac{2pr}{p^2 + q^2 + r^2}, \qquad n_1 = \frac{2qr}{p^2 + q^2 + r^2},$$

where p, q, and r are any three integers.

Having so selected l_1, m_1, and n_1, we have next to solve, in rational numbers, the simultaneous equations

$$\begin{cases} l_2^2 + m_2^2 + n_2^2 = 1, \\ l_1l_2 + m_1m_2 + n_1n_2 = 0. \end{cases} \tag{2}$$

If we eliminate l_2 from these equations, the resulting equation can be put into the form

$$\left\{m_2 + \frac{m_1n_1n_2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1n_2}{l_1^2 + m_1^2}\right\}^2 = \left\{\frac{l_1^2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1m_1}{l_1^2 + m_1^2}\right\}^2, \tag{3}$$

since the second member may be written as $l_1^2/(l_1^2 + m_1^2)$. Two solutions of

equation (3) are evidently given by

$$\begin{cases} (l_1^2 + m_1^2)m_2 + m_1n_1n_2 = l_1^2, \\ l_1n_2 = \pm\, l_1m_1, \end{cases}$$

and

$$\begin{cases} (l_1^2 + m_1^2)m_2 + m_1n_1n_2 = \pm\, l_1m_1, \\ l_1n_2 = l_1^2. \end{cases}$$

From the first of these two solutions and the second of equations (2)

$$l_2 = \frac{-\, l_1m_1(1 \pm n_1)}{l_1^2 + m_1^2}, \qquad m_2 = \frac{l_1^2 \mp m_1^2n_1}{l_1^2 + m_1^2}, \qquad n_2 = \pm\, m_1,$$

and from the relations

$$l_3 = m_1n_2 - m_2n_1, \qquad m_3 = n_1l_2 - n_2l_1, \qquad n_3 = l_1m_2 - l_2m_1, \tag{4}$$

mentioned above, we have

$$l_3 = \frac{-\, l_1^2n_1 \pm m_1^2}{l_1^2 + m_1^2}, \qquad m_3 = \frac{-\, l_1m_1(n_1 \pm 1)}{l_1^2 + m_1^2}, \qquad n_3 = l_1.$$

Similarly, from the second of our two solutions and the second of equations (2)

$$l_2 = \frac{-\, l_1^2n_1 \mp m_1^2}{l_1^2 + m_1^2}, \qquad m_2 = \frac{-\, l_1m_1(n_1 \mp 1)}{l_1^2 + m_1^2}, \qquad n_2 = l_1,$$

and with (4)

$$l_3 = \frac{l_1m_1(1 \mp n_1)}{l_1^2 + m_1^2}, \qquad m_3 = \frac{-\, l_1^2 \mp m_1^2n_1}{l_1^2 + m_1^2}, \qquad n_3 = \pm\, m_1.$$

A more general solution of equation (3) is given by

$$\begin{cases} (l_1^2 + m_1^2)m_2 + m_1n_1n_2 = sl_1^2 \mp tl_1m_1, \\ l_1n_2 = tl_1^2 \pm sl_1m_1, \end{cases} \tag{5}$$

where s and t are any two rational numbers satisfying the equation

$$s^2 + t^2 = 1.$$

Then s and t can always be expressed in the form

$$s = \frac{1 - k^2}{1 + k^2}, \qquad t = \frac{2k}{1 + k^2},$$

k any rational number (or $k = \infty$, which gives $s = -1, t = 0$). From equations (5) and the second of equations (2) we have, finally, the one-parameter family,

$$\begin{cases} l_2 = \dfrac{-\, sl_1m_1 \pm tm_1^2 - tl_1^2n_1 \mp sl_1m_1n_1}{l_1^2 + m_1^2}, \\ m_2 = \dfrac{sl_1^2 \mp tl_1m_1 - tl_1m_1n_1 \mp sm_1^2n_1}{l_1^2 + m_1^2}, \\ n_2 = tl_1 \pm sm_1, \end{cases}$$

equations (4) giving the direction cosines of the remaining line, viz.,

$$\begin{cases} l_3 = \dfrac{tl_1m_1 \pm sm_1^2 - sl_1^2n_1 \pm tl_1m_1n_1}{l_1^2 + m_1^2}, \\ m_3 = \dfrac{-sl_1m_1n_1 \pm tm_1^2n_1 - tl_1^2 \mp sl_1m_1}{l_1^2 + m_1^2}, \\ n_3 = sl_1 \mp tm_1. \end{cases}$$

The solution last given is the most general rational solution possible; for evidently any other solution must correspond to a solution of equation (3) more general than (5). To show that (5) is the most general solution, let us simplify the notation and consider the most general rational solution, for x and y, of the equation

$$x^2 + y^2 = a^2 + b^2, \qquad \text{or} \qquad x^2 - a^2 = -y^2 + b^2.$$

Then

$$x + a = \frac{1}{k}(y - b), \qquad x - a = -k(y + b),$$

or

$$x = \frac{(1 - k^2)a - 2kb}{1 + k^2}, \qquad y = \frac{2ka + (1 - k^2)b}{1 + k^2}.$$

Hence we have determined the most general set of these mutually perpendicular lines each of which has rational direction cosines.

A NOTE ON THE PROBLEM OF THE EIGHT QUEENS.

By W. H. BUSSEY, University of Minnesota.

Finite geometries were defined by Veblen and Bussey in the *Transactions of the American Mathematical Society*, volume 7 (1906), pp. 241–259. References to existing literature of the subject were given in this MONTHLY, *1921*, 85–86. The simplest case of a finite plane geometry based upon an odd prime, the euclidean plane geometry, modulo 3, was presented in detail by Bennett, in this MONTHLY, *1920*, 357–361.

The Problem of the Eight Queens is the determination of the number of ways in which eight queens can be placed on a chess board—or, more generally, in which n queens can be placed on a square board of n^2 cells—so that no queen can take any other. It was proposed originally by Franz Nauck.[1]

The object of this note is to show that in the special case in which n is a prime number p there is a connection between the problem of the queens and the lines of the finite plane geometry of p points to the line.

The cells of the chess board are represented by their middle points which

[1] For the history of the problem see Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chapter 9. A brief discussion of the problem and its solution is given by Ball, *Mathematical Recreations and Essays*, fifth, sixth or seventh edition, pp. 113–118.

constitute a finite euclidean plane geometry of p points to the line. In order that one queen may take another, the two queens must occupy cells whose representative points are on a vertical line, on a horizontal line, on a line of slope 1, or on a line of slope $p - 1$. (The slope $p - 1$ is the same as the slope -1 since $p - 1 \equiv -1$, modulo p.)

Therefore, if p queens be placed on the cells whose representatives are the points of a line whose slope is any one of the integers $2, 3, 4, \cdots, p - 2$, no queen can take any other queen. The total number of such lines, each of which furnishes a solution of the problem of the p queens, is $p^2 + p - 4p$ or $p^2 - 3p$. When $p = 5$, the number of solutions obtained by this method is 10, which happens to be all the solutions that exist. When $p = 7$, the number of solutions furnished by this finite geometry method is 28; but, as a matter of fact, there are 40 solutions[1] when $p = 7$. For higher values of p, the lines of the finite geometry furnish some, but not all, of the solutions of the problem.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

25. Montucla's Closing Years.

The little that we know about the great men of our varied spheres of interest and the careless surmises that we make as to their lives are evident whenever we look even slightly below the surface that lies open to the world. Anyone who reads an ordinary biographical sketch of Montucla, for example, gains an impression of a man who was born, who wrote the first great history of mathematics, and who died in the fullness of years.[2] In an earlier article in this series[3] I have called attention to a little side-light thrown upon his life by a note from his collaborator Lalande. It seems worth the effort, however, to call further attention to his closing years by publishing a portion of a letter, now in my collection and written to some unnamed friend, which gives a nearer view of these last years of one whose portraits show as a well-fed "gentleman of the old school," bland, placid, content with the world, and appreciative of the praise that this same world had bestowed upon him.

The letter is garrulous to the point of being wearisome, but a portion will suffice to give a picture of a poor, harassed, discouraged old man, suffering in mind and body, neglected by his friends and deserted by his family,—a subject for the pity of a world not then given to pitying anyone, and of a later world

[1] Cf. Ball, *l.c.*, p. 116.

[2] Jean Étienne Montucla, born September 5, 1725; died at Paris, December 18, 1799. He was married at Grenoble in 1763. The first edition of his history appeared in 1758. It is a work of great impartiality and erudition. By the Revolution he lost his position in the civil service and was left with no means of livelihood. In 1794 he secured a place under the revolutionary government and he finally, four months before his death, secured a small pension of 2,400 francs.

[3] This Monthly, *1921*, 207.

which has well-nigh forgotten Montucla, the man. The letter was written on October 12, 1798, not many months before his death, and is in part as follows:

Tonnerre 20. vend.re 6.[1]

I have delayed an answer, my dear old friend, to your latest, in the hope that I might reply by word of mouth. I have now decided to send it in writing, traveling not having been very agreeable during the summer. I shall therefore follow your letter by article.

Not being able to get a word out of M. Denizet [2] I asked Madame Denoust [3] directly, and then through the mediation of the divine Lalande,[4] to stir him up. Madame Denoust replied: "It would be useless for me to take the matter up with my brother-in-law[5] and it would be necessary for me to go to Paris to take charge of the matter directly." The "Divine One" [6] did more than that, for he offered me a bed and some money, and let M. Denizet know about it.

M. Denizet said to Madame Denoust that her family had become bankrupt and told the divine one that he would only be getting her into trouble if he encouraged her to come to Paris. And then he suddenly wrote me a long letter.

Since the divine one told me that M. Denizet did not even read my letters to him, I answered him briefly and a few days later I received four lines from him. I was waiting for the result of his promises when Demottier, "the Picard"—my old coachman, wrote to me that I had become a widower and that my wife[7] had died at the abbé's at Gournai, near Compiègne.

A few weeks afterwards M. Denizet came to Tonnerre and told me that an Abbé[8] Carron had taken charge of the business affairs of my three daughters and that he was going to file a claim against my pension,—a pension which had not been paid since January, 1794. I offered to write to my eldest daughter, and, although the initiative should have come from her, I took the first step.

On the 15th of August M. Denizet wrote to me that he had obtained a pension of 800 livres, of which the first installment of 400 livres was paid immediately, that the second one of 400 livres would be paid on January first, and that he would take the matter up again,—this being looked upon only as a provisional payment.

I replied that I owed some money for my board at Tonnerre during the three years preceding and that I hoped he would carry out his plans.

As to my eldest daughter, she sent me a cool letter[9] from which I could see that she had been influenced against me. Indeed, is it possible to believe that this letter which I was compelled to write to her was the first that had passed between us since she was born!

These 400 livres paid on August 15 have not yet reached me because Madame Denizet, intending to take the trouble to send them to me, had received them all right, but being away on the 19th Fructidor,[10] the banks were closed and she used my money to come to Tonnerre, giving me the assurance that her husband would pay it to me in October. M. de Denizet has not come and I have just written to him to let me have at least a mouthful,—the only assistance that has been given me since January first, 1794. He has replied that I shall receive it on the 26th.

Moreover, M. Denizet, during his brief sojourn at Tonnerre last summer, told me that I might go to Paris when I had the means and that he saw no objection to my so doing. In anticipation of this I took the matter up with several people, but the 18th Fructidor[11] has upset everything and I do not know what will become of it all.

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[1] October 12, 1798. Tonnerre is about 80 miles southeast of Paris.

[2] The names in the letter are mostly those who left no impression upon the world and are now forgotten. M. Denizet was apparently connected with the bureau of civil pensions.

[3] Evidently the sister-in-law of M. Denizet.

[4] Joseph Jérôme Le Francais de Lalande, born at Bourg-en-Bresse, July 11, 1732; died at Paris, April 4, 1807. Montucla speaks of him ironically as "the Divine One," apparently from his manner.

[5] That is, by correspondence with M. Denizet.

[6] Capital, *Divin.*

[7] A joke, he and his wife not being on good terms. His wife survived him.

[8] Aé in the original.

[9] Une lettre respectuesement sèche.

[10] September 6. The *coup d'état* of Barras, Larévellière, and Reubell had taken place the day before and everything was at a temporary standstill.

[11] September 5, the day of the *coup d'état* already mentioned.

During his sojourn at Tonnerre M. Denizet saw Madame Quingeri and Madame Leprince, two frequent visitors there, saying in a lofty way that he never forgot his old friends.

The day of his departure he finished up the business and asked me to dinner. He was very affable and, I must say in justice to him, he paid me special attention. I do not know whether, with patience, I shall ever see the end of this business. If I do, I shall need to have shown a lot of it.

I now come to the second part of your letter.

I have always blamed the misplaced pride of Bailly[1] and Condorcet.[2] As to Duséjour[3] I was under the impression that he had been executed, but you tell me that he died of fear of terrorism.[4]

. . . I was right about complaining about Madame de Marcheval and Madame de Lesseville.[5] These two females[6] have never ceased to stir up my wife against me. I made a short call at Auteuil in August 1779[7] and took supper at my aunt's, but I did not succeed in making any better impression. She had already got fixed in my wife's head the necessity for leaving me and managed to obtain the *lettre de cachet*,[8] on the 30th of the following November, through her friend Amelot. Now talk of gratitude among relations, especially after all that I had done for them!

The rest of the letter relates to suggested changes in Montucla's history, then being revised by Lalande, and need not concern us. Enough has been given to allow us to picture to ourselves a very human man, enduring very human ills, and approaching his end with very human complaints,—a man the exact antithesis of what his contemporary conventional portraits show to the world at large.

QUESTIONS AND DISCUSSIONS

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

REPLIES.

36, 1. [*1919*, 69].[9] For what values of n can $\cos(2\pi/n)$ be expressed in the form $(a + \sqrt{b})/c$ where a, b and c are integers?

I. REPLY BY R. S. UNDERWOOD, Alabama Polytechnic Institute.

Using the formula[10]

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \frac{n(n-4)(n-5)}{3!}(2\cos\theta)^{n-6} + \cdots,$$

we get

$$2 = (2\cos 2m\pi/n)^n - \frac{n}{1}(2\cos 2m\pi/n)^{n-2} + \cdots;$$

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[1] Jean Sylvain Bailly, born at Paris in 1736, executed at Paris in 1793. Arago's biography is well known. The misplaced pride consisted in standing for moderate principles against a bloodthirsty mob. Both he and Condorcet were prominent as mathematicians and as statesmen.

[2] Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet, born at Ribemont in 1743; died at Bourg-la-Reine in 1794, a suicide by poison, in prison, to prevent being taken to Paris for execution.

Tallentyre wrote of him: "Since he never gave himself blindly to any one faction, all factions have distrusted and condemned him. To the Royalist he is a Revolutionist; to the Revolutionist he is an aristocrat. The thinker cannot forgive him that his thought led him to deeds and words; the man of action cannot forget that he was a thinker and dreamer to the end.

[3] Achille Pierre Dionis du Séjour (1734–1794), a celebrated astronomer. He fled to the country during the Terror, concealed himself, and died there.

[4] The recipient of the letter (apparently) has added the words, "He died of fear of death."

[5] The Marquis de Lesseville, now living at Châlons sur Marne, tells me that this was probably his great grandmother. The family is an old one.

[6] Femelles, female beasts, a term of great contempt.

[7] This shows that for the last twenty years of his life Montucla's family life was unpleasant. The event was only sixteen years after his marriage.

[8] Order for arrest.

[9] The other parts of this question were answered by the proposer, Professor Harris Hancock, *1919*, 292–295.

[10] See Problems—Notes 3, *1921*, 38.

so that the values of $2\cos(2m\pi/n)$ for $m = 1, 2, \cdots, n$ are the roots of the equation

$$\left\{ x^n - nx^{n-2} + \frac{n(n-3)}{2} x^{n-4} - \cdots \right\} - 2 = 0.$$

Since the coefficients in this equation are integers, the coefficients k and k_1 of any rational quadratic factor $x^2 + kx + k_1$ must also be integers. We may reject values of k and k_1 which make the roots of $x^2 + kx + k_1 = 0$ imaginary or greater than 2 in absolute value. Furthermore all the n roots of the original equation will be needed for the values of $2\cos(2m\pi/n)$ when $m = 1, 2, \cdots, n$; hence neither one of the roots of the quadratic can be greater than 2.

With this restriction, the possible irreducible quadratic factors are limited to[1] $x^2 - 2$, $x^2 - 3$, and $x^2 \pm x - 1$, while the only possible rational linear factors are x, $x \pm 1$, and $x \pm 2$. Hence the only possible rational and quadratic surd values for $\cos(2m\pi/n)$ are 0, $\pm 1/2$, ± 1, $\pm \sqrt{2}/2$, $\pm \sqrt{3}/2$, and $(\pm 1 \pm \sqrt{5})/4$; and the only values of n for which $\cos(2\pi/n)$ is of the form $(a + \sqrt{b})/c$, including the case $b = 0$, are 1, 2, 3, 4, 5, 6, 8, 10, and 12.

II. Remarks by the Editor.

It is evident that Professor Underwood's method may be used to find the positive integers n for which $\cos(2\pi/n)$ satisfies an equation of given degree k with integral coefficients. It is necessary that $2\cos(2\pi/n)$ satisfy an equation of the form

$$x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots = 0, \tag{1}$$

in which (*a*) $a_1, a_2, \cdots$ are integers, and (*b*) all roots are real and not greater than 2 in absolute value. On account of (*b*), $|a_1| \leqq 2k$, $|a_2| \leqq 4\binom{k}{2}$, etc. Hence the number of equations (1) satisfying (*a*) and (*b*) is finite, and the possible equations may be examined in succession.

It will be noticed that in the cases $k = 1$ and $k = 2$ all "possible" equations led to actual solutions, and it might be thought that in the higher cases some difficulty would occur in recognizing those equations which, while conforming to (*a*) and (*b*), failed to satisfy the trigonometric test. Such difficulty however would not occur in view of the following interesting theorem, due to Kronecker:[2]

If an equation (1) *satisfies* (*a*) *and* (*b*), *each of its roots is of the form* $2\cos(2m\pi/n)$, *where* m *and* n *are integers.*

For proof, let $x = 2\cos\theta$, and let $y = 2\cos 2\theta = x^2 - 2$. On eliminating x between this equation and (1) by any of the ordinary methods, we get an equation in y of type (1) which is seen to satisfy (*b*). From the nature of the transformation it also satisfies (*a*). Hence, by applying successively transformations $z = y^2 - 2$, $u = z^2 - 2$, etc., we obtain always equations of type (1) satisfying (*a*) and (*b*). But such equations are finite in number. Therefore eventually we obtain two that are identical, and have the same roots. Now it may be that the roots of these two equations do not correspond to themselves, but are permuted by the transformations. In such a case we may apply more transformations until the permutation has been made $k!$ times, when every root will return to its own position. It will therefore happen that after, let us say, p and q transformations any root $2\cos\theta$ of (1) will be transformed into the same number. Hence $2^p\theta = 2r\pi \pm 2^q\theta$, where r is an integer and $p \neq q$; and therefore θ is commensurable with 2π.

The method used hitherto would prove very laborious for values of k exceeding 2 or 3; but the question may be settled in another way by means of the theory of the primitive nth roots of unity. If the prime divisors of n are $p, q, \cdots$, the number of primitive roots of $x^n = 1$ is $\varphi(n) = n(1 - 1/p)(1 - 1/q)\cdots$, being the number of positive integers prime to and not greater than n. They satisfy an equation $f(x) = 0$ of degree $\varphi(n)$, having integral coefficients. If n is greater than 2, $\varphi(n)$ is even, and the coefficients of $f(x)$ are unaltered by writing in reversed order; so that the transformation $y = x + x^{-1}$ leads to an equation in y of degree $\frac{1}{2}\varphi(n)$. But one of the roots in x is $e^{2i\pi/n}$; and therefore one of the roots in y is $2\cos(2\pi/n)$. It follows that $\cos(2\pi/n)$ is a root of a rational equation of degree $\frac{1}{2}\varphi(n)$.

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[1] The question of rational linear factors (and therefore of rational cosines) was discussed by Professor Underwood in a recent issue of this Monthly (*1921*, 374).—Editor.

[2] "Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten", *Journal für die reine und angewandte Mathematik*, vol. 53, p. 173; *Werke*, vol. 1, 1895, p. 107. Kronecker deduced the present theorem from another concerned with roots of unity; but his method is essentially similar to that of the direct proof given above.

The question now is whether $\cos(2\pi/n)$ can be a root of a rational equation of lower degree. Let it be a root of an equation of degree k. The substitution of the exponential form for the cosine then shows that $e^{2i\pi/n}$ is a root of a rational equation of degree $2k$. But the equation of primitive nth roots of unity is not rationally reducible;[1] that is, $e^{2i\pi/n}$ is not a root of any rational equation of degree less than $\varphi(n)$. Therefore k is not less than $\frac{1}{2}\varphi(n)$. We have then the theorem: *If n is an integer greater than 2, $\cos(2\pi/n)$ is a root of an irreducible equation of degree $\frac{1}{2}\varphi(n)$.*

The simpler cases are:

$\varphi(n) = 2$ when $n = 3, 4, 6$; $\cos(2\pi/n)$ rational, as also for $n = 1, 2$.
$\varphi(n) = 4$ when $n = 5, 8, 10, 12$; $\cos(2\pi/n)$ a quadratic surd.
$\varphi(n) = 6$ when $n = 7, 9, 14, 18$; $\cos(2\pi/n)$ a cubic surd.
$\varphi(n) = 8$ when $n = 15, 16, 20, 24, 30$; $\cos(2\pi/n)$ a quartic surd.

DISCUSSIONS.

Professor Hathaway obtains an integral reduction formula which includes as special cases the formulas usually given in works on the integral calculus.[2] In a note following the paper it is shown how the formula may be regarded as a transform of one of these special cases. This, of course, does not prevent it from being also a generalization.

A General Type of Reduction Formula.

By A. S. Hathaway, Rose Polytechnic Institute.

Let X, Y, Z be functions of a single variable, such that

$$AX^2 + BY^2 + CZ^2 = 0, \tag{1}$$

A, B, C being constants. By differentiation,

$$AXdX + BYdY + CZdZ = 0; \tag{2}$$

and from (1) and (2)

$$\frac{YdZ - ZdY}{AX} = \frac{ZdX - XdZ}{BY} = \frac{XdY - YdX}{CZ} = dT \text{ (for brevity).} \tag{3}$$

The general integral considered is

$$\beta(L, M, N) = \int X^L Y^M Z^N dT, \text{ where } L + M + N + 1 = 0. \tag{4}$$

The integral is homogeneous of order zero. It is not altered by substituting for X, Y, Z any variable common multiples of them, VX, VY, VZ. These also satisfy (1) and (3).

Further the integral is not altered by permuting X, Y, Z, concurrently with A, B, C and L, M, N; except that the sign is changed when the permutation is not cyclic (from the definition of dT). This means that any formula of expansion of the integral in powers of two bases is equivalent to six different formulas

[1] Dedekind's proof for the general case (n composite) is given in H. Weber, *Lehrbuch der Algebra*, volume 1 (Braunschweig, 1898), p. 596, or *Traité d'Algèbre Supérieure* (French translation by J. Griess, Paris, 1898), p. 636. For proof by Arndt, see P. Bachmann, *Die Lehre von der Kreistheilung* (Leipzig, 1872), p. 38.

[2] W. A. Granville, *Elements of the Differential and Integral Calculus*, Boston, 1911, pp. 350-360.

obtained by the six different permutations of concurrent parts. We propose to establish such a formula, and show the great variety of its applications.

We have $d(X^{L+1}Y^{M-1}Z^{N+1}) = X^LY^{M-2}Z^N[(L+1)YZdX + (M-1)ZXdY + (N+1)XYdZ] = X^LY^{M-2}Z^N[(M-1)CZ^2 - (N+1)BY^2]dT.$

Integrating and solving for $\beta(L, M, N)$,

$$\beta(L, M, N) = \frac{X^{L+1}Y^{M-1}}{B(-N-1)Z^{-N-1}} + \frac{-C(M-1)}{B(-N-1)}\beta(L, M-2, N+2). \quad (5)$$

The reduced integral is of the same form, since one exponent is increased as much as the other is decreased; and (5) applies to it, with proper change of exponents, M to $M-2$, N to $N+2$. This can be continued in general indefinitely. The resulting formula[1] is:

$$\int X^LY^MZ^NdT = \frac{X^{L+1}Y^{M-1}}{B(-N-1)Z^{-N-1}} + \frac{-C(M-1)X^{L+1}Y^{M-3}}{B^2(-N-1)(-N-3)Z^{-N-3}} + \cdots + \frac{(-C)^{k-1}(M-1)(M-3)\cdots(M-2k+3)X^{L+1}Y^{M-2k+1}}{B^k(-N-1)(-N-3)\cdots(-N-2k+1)Z^{-N-2k+1}} + \frac{(-C)^k(M-1)\cdots(M-2k+1)}{B^k(-N-1)\cdots(-N-2k+1)}\int X^LY^{M-2k}Z^{N+2k}dT. \quad (6)$$

If M were an odd positive integer, we would have an algebraic integral of k terms, where $k = (M+1)/2$, since a factor zero appears in the coefficient of the reduced integral. We could not carry the reduction so far, however, if at the same time N were an odd negative integer not numerically larger than M, as a zero factor would appear first in the denominator, and the series must be stopped before that occurs. The effect of a possible zero factor in stopping reduction may be thus stated: *an odd exponent cannot be reduced so that its sign changes.* If positive, it must stop at 1; if negative, at -1. We have: *the integral $\beta(L, M, N)$ is algebraic in X, Y, Z when (and only when) one exponent is an odd*

[1] Equation (6) may be written:

$$\beta(L, M, N) = X^{L+1}\left\{\frac{-C\cdot M-1\cdot Y}{B\cdot -N-1\cdot Z}k\right\} + \left\{\frac{-C\cdot M-1}{B\cdot -N-1}k\right\}\beta(L, M-2k, N+2k)$$

The bracketed terms are "indices" for corresponding sums and products, and k is the number of integrated terms.

By permuting, Y and Z become *any two of the variables*, M and N, their exponents, B and C, their coefficients in (1), *with the factor* -1 *before that coefficient which follows the other in the cyclic order ABCA.*

In this notation, the illustrative example given later is,

$$\beta(4, 6, -11) = X^4\left\{\frac{3\cdot 5\cdot Y}{2\cdot 10\cdot Z}3\right\} + \left\{\frac{3\cdot 5}{2\cdot 10}3\right\}\beta(4, 0, -5).$$

$$\beta(4, 0, -5) = Y\left\{\frac{-3\cdot 3\cdot X}{-7\cdot 4\cdot Z}2\right\} + \left\{\frac{-3\cdot 3}{-7\cdot 4}2\right\}\beta(0, 0, -1).$$

A complete reduction may therefore be written in this notation, given variables, exponents, and coefficients. Preferably reduce the two *numerically greatest* exponents, one of which is positive, the other negative, by (4).

positive integer, the other two being any numbers subject to (4), *rational or irrational, that are not both negative odd integers.*

When one exponent is a positive odd integer and the other two negative odd integers, the integration may be made with a logarithmic term. For the exponents reduce to 1, − 1, − 1; and

$$\int \frac{XdT}{YZ} = \frac{1}{A}\int\left(\frac{dZ}{Z} - \frac{dY}{Y}\right) = \frac{1}{A}\log\frac{Z}{Y}.$$

Hence $\beta(L, M, N)$ *can always be integrated when one exponent is a positive odd integer.*

The integration is possible if one exponent be an odd negative integer and the other exponents rational numbers. For integral exponents (excluding an odd positive one, already considered), by (4), two must be even and one odd negative, so that the exponents reduce to − 1, 0, 0. In this case, $\int dT/X = \int(YdZ - ZdY)/(AX^2) = -\int(YdZ - ZdY)/(BY^2 + CZ^2) = \int dx/(Bx^2 + C)$, where $x = Y/Z$ (an anti-tangent or logarithm as the signs of B, C are alike or not).

With one odd negative and two fractional exponents, the exponents reduce to − 1, m/n, − m/n, by (4), and we have the preceding form multiplied by $x^{m/n}$ ($x = Y/Z$), which integrates as a rational fraction by the substitution $x = z^n$.

It seems probable that $\beta(L, M, N)$ is not integrable in elementary functions when no exponent is an odd integer. By (4), there must be an odd exponent when all are integers.

In reducing by (6), it is first necessary to determine bases, coefficients, and exponents. Powers of X, Y, Z must be factors of the given differential to be integrated, their relation (1) being found by inspection. Then dT is computed by (3), and the quotient of the differential by dT must be, to a constant factor, $X^L Y^M Z^N$, where $L + M + N = -1$. If there are only two functions X, Y, with a relation $AX^2 + BY^2 + C = 0$, then $Z = 1$, $dT = dX/BY$, and the relation (4) is no restriction, but only a determination of the exponent of $Z = 1$, required in (6).

Of the six possible permutations of formula (6), only two can be used to reduce exponents to their smallest values (between 1 and − 1). Namely, there must be a positive exponent in place of M, and a negative one in place of N (only two, since by (4) two exponents are positive and one negative, or vice versa, or all are negative and reduced). Generally either pair of exponents may be used to reduce its numerically smallest exponent (leaving the third unchanged), and in the reduced integral there is only one pair for further reduction, and this pair leaves the exponent just reduced unchanged.

For example, find $\int x^4(3 + 4x^2)^{5/2}(2 + 5x^2)^{-6}dx$. The variables are $X = x$, $Y = (3 + 4x^2)^{1/2}$, $Z = (2 + 5x^2)^{1/2}$, with $7X^2 + 2Y^2 - 3Z^2 = 0$. $dT = dx/(YZ)$, and the given integral is therefore $\int X^4 Y^6 Z^{-11} dT = \beta(4, 6, -11)$, since $4 + 6 - 11 + 1 = 0$.

To apply (6) we have a choice between the pairs of exponents (4, − 11) and

(6, − 11). Taking the latter, we have

$$\beta(4, 6, -11) = X^5\left[\frac{Y^5}{2\cdot 10\cdot Z^{10}} + \frac{3\cdot 5Y^3}{2^2\cdot 10\cdot 8Z^8} + \frac{3^2\cdot 5\cdot 3Y}{2^3\cdot 10\cdot 8\cdot 6Z^6}\right] + \frac{3^3\cdot 5\cdot 3\cdot 1}{2^3\cdot 10\cdot 8\cdot 6}\beta(4, 0, -5).$$

For next reduction we have only the pair (4, − 5).

$$\beta(4, 0, -5) = Y\left[\frac{X^3}{-7\cdot 4Z^4} + \frac{-3\cdot 3X}{7^2\cdot 4\cdot 2Z^2}\right] + \frac{3^2\cdot 3\cdot 1}{7^2\cdot 4\cdot 2}\beta(0, 0, -1),$$

$$\beta(0, 0, -1) = \int dT/Z = \int \frac{XdY - YdX}{-3Z^2} = \int \frac{XdY - YdX}{7X^2 + 2Y^2} = \frac{1}{\sqrt{14}}\tan^{-1}\frac{\sqrt{2}Y}{\sqrt{7}X}.$$

The solution is complete on substituting values found.

Note the following particular form:

$$\tfrac{1}{2}\int (a + bx)^{(L-1)/2}(a' + b'x)^{(M-1)/2}(a'' + b''x)^{(N-1)/2}dx = \beta(L, M, N), \qquad (L + M + N + 1 = 0).$$

$$X^2 = a + bx, \qquad Y^2 = a' + b'x, \qquad Z^2 = a'' + b''x,$$
$$(a'b'' - a''b')X^2 + (a''b - ab'')Y^2 + (ab' - a'b)Z^2 = 0.$$
$$dT = dx/(2XYZ).$$

Of other forms, note:

$$\int \sin^L x \cos^M x\, dx; \qquad X = \sin x, \qquad Y = \cos x, \qquad Z = 1,$$
$$X^2 + Y^2 - Z^2 = 0, \qquad dT = dx.$$
$$\int \tan^L x \sec^{M+1} x\, dx; \qquad X = \tan x, \qquad Y = \sec x, \qquad Z = 1.$$
$$-X^2 + Y^2 - Z^2 = 0, \qquad dT = \sec x\, dx.$$

In each of these the exponent of Z is $N = -L - M - 1$.

NOTE BY THE EDITOR.

In the application of the reduction formula, the problem is to recognize the four variables X, Y, Z, and T. It therefore suggests itself that we should look a little into the meaning of these variables in the integral. The equations (1) and (3) show that only two of the four are really independent, so that the formula in question ought to be transformable into a canonical form containing only one arbitrary function. But actually the situation is simpler still, on account of the homogeneity to which the author has alluded, which makes the ratios of X, Y, Z the only functions of importance.

In dealing with real quantities, we must be able to throw (1) into some such form as $(Z/c)^2 = (X/a)^2 + (Y/b)^2$. We may then use two variables Z and θ, so that $X = (a/c)Z\cos\theta$, and $Y = (b/c)Z\sin\theta$. (Geometrically, we are dealing with points (X, Y, Z) on a quadric cone, and θ is the eccentric angle in a principal elliptic section.) It follows that $dT = c^2(XdY - YdX)/Z = abZd\theta$.

Now it must be supposed that we integrate along some path lying on the cone and given by an equation $Z = F(\theta)$. But the homogeneity principle prepares us to see Z disappear from the integral, as indeed it does, the result of the transformation being $\int X^L Y^M Z^{-L-M-1} dT = a^{L+1} b^{M+1} c^{-L-M} \int \cos^L \theta \sin^M \theta \, d\theta$.

Thus every integral of the type in question is transformable into this one particular form. In the author's first example, the required transformation is

$$\cos \theta = \sqrt{\frac{7}{3}} \cdot x(2 + 5x^2)^{-1/2}.$$

It may be added that, if we possess the general reduction formula developed above, nothing is gained by this transformation from the point of view of integration. In fact the reduction formula is exactly the same before and after transformation.

RECENT PUBLICATIONS.

REVIEWS.

The Principles of Geometry. By H. F. BAKER. Volume 1, Cambridge University Press, 1922. 182 pp. Price 12 shillings.

That prince of teachers, the late Jules Tannery, once wrote in the introduction to a book by one of his former pupils: "Un petit livre est rassurant."

When a book is not only short but well printed, with wide margins and in a flowing style of the King's English, the reassuring impression is much strengthened. But the reader who takes up the work before us under these pleasant impressions, with the idea that in spite of a lack of any special preparation in the way of familiarity with the subject matter and point of view, he is going in a few hours to reach the real substance, this reader will have a very prompt chance to guess again.

It will take a much greater convulsion than the late World War to stop the output of books on Projective Geometry, especially from English writers. Yet so far, in England, they have clung to the tradition of Cremona and Stephen Smith, taking the metrical definition of cross ratios as fundamental, regardless of the fact that Klein proved half a century ago that the cross ratio can be reached by purely descriptive processes, and that Continental and American writers have been doing this in their texts for a generation. Sooner or later some Englishman was bound to fall into line, and it is a subject for satisfaction that the first should be such a distinguished scholar as Baker. The reason for the delay is doubtless this, that English boys are drilled in a subject of which Americans have scarcely heard the infelicitous name, Geometrical Conics. The metrico-projective treatment flows from this in the most natural manner. Moreover the present writer does not say that he is actually writing a book on projective geometry, and he starts his second chapter with a discussion of the descriptive properties of a limited region. But as soon as he has strengthened this to the point where it can stand on its legs, he adjoins to the universe of discourse ideal,

or, as he says, "postulated" points, until he has the whole projective domain once more available.

The book has one most unusual characteristic, informality. The writer says in the fourth line of the preface: "It (the book) assumes those relations of position for points, lines and planes which, furnished with a pencil, a ruler, some rods and some string, a student may learn by drawing diagrams and making models."

There is no question of the independence of axioms: the writer states what he presupposes in the following fashion (pp. 4, 5): "Through an arbitrary point there pass an infinite number of lines, of which one passes through any other arbitrary point. Thus a line is determined uniquely by any two points. The line contains an infinite number of points besides these two determining ones, and is determined by any two of these. Through any given line there pass an infinite number of planes, of which one passes through any point not lying on the given line. Thus a plane is determined uniquely by any three points which do not lie in line. The plane contains an infinite number of points besides the determining ones, and is determined by any three of these which do not lie in line. The plane entirely contains the line which is determined by any two of its points. Thus if two lines have a common point, and we take two other points, one on each of the lines, the plane determined by the three points contains both the lines, that is, two lines with a common point determine a plane, containing both lines. It is also true that two lines which are in the same plane have a common point. We have said that there is an infinite number of planes, all containing the points of a given line; it is also true that any two planes have common points lying on a line, or that any two planes intersect in a line."

This casual method of statement is entirely characteristic. The writer seems to have an ingrained dislike for type display. He will head a long paragraph "So and So's Theorem" and put the actual statement, with or without italics, at such point in the argument as pleases him. The usual assortment of labels, Axiom, Postulate, Assumption, Restriction, Theorem, he calmly ignores. At times one is tempted to resent this, and exclaim irritably: "Why in the name of common honesty doesn't the good man tell us clearly once for all what he assumes, and what he proves?"

Fair and softly, dear reader, fair and softly, you won't get anywhere by storming at him. If you realize that this is a thoroughly scholarly work, and read every word, you will find that everything necessary is there. If you mistake it for a newspaper, and try to read nothing but the headlines, that is not his fault.

Here is a case in point. On p. 11 we find the usual quadrilateral construction for the harmonic point-set. A wicked gleam will come into the eye of the malevolent reader, as he exclaims: "Aha, how do we know that this yields anything new? How do we know that the diagonals of a complete quadrilateral are not concurrent? Does the writer expect us to use pencils and rods and string?"

Not in the least; when he gets good and ready, namely three pages later, he calmly says: "But we shall, in accordance with a general assumption referred to in the beginning, assume provisionally that the point D does not coincide with the point C."

The first chapter deals with the general projective geometry, where no mention is made of order, or any equivalent characteristic of the real domain. We have the usual concurrence, collinearity and coplanarity theorems, which the writer well describes as "theorems of incidence." A good deal of attention is paid to "related ranges" of points. This means two ranges that are in perspective with the same third range. It is shown that two ranges which are related to a third are related to one another. Thus, when two one-dimensional forms are the first and last in a long string of projections and intersections, they may be connected by a single piece of machinery with only two joints. Next, of course, comes the fundamental theorem of projective geometry, whereby the projective relation between two one-dimensional forms is completely determined by the fate of three elements. The writer turns this over in every possible way, showing that it is equivalent to any one of four different theorems. The one which he prefers, and which constitutes the central assumption for this whole chapter, is Pappus's theorem, or Pascal's, where the conic is two lines. The preference for this particular form of statement is characteristic of the present work. This discussion of the reduction of perspectivities and the fundamental theorem is very careful; truth compels us to add that it is not short; the two require twenty-one pages.

All writers on the foundations of geometry, sooner or later, draw on arithmetic for aid; the present writer does so at the point we have now reached. Large letters are used to denote points, small letters denote a system of numbers about which we make all of the usual assumptions for addition and multiplication, *except the commutative law for the latter.* If the point P be defined as the same as the point rP, then the points of the line determined by two given points will be the system of points linearly dependent on them in the present symbolism, and in fact, it is easily shown that the present system of linear dependence completely covers the former theorems of incidence.

The correlation of arithmetical and geometrical operations is next developed, very satisfactorily, on the following lines:

(*a*) The relation between the numbers r and $-r$. The points $A + rB$ and $A - rB$ shall be harmonic conjugates with regard to the points A and B.

(*b*) The relation between the numbers r, s, and $r + s$. The point B shall be one vertex of a complete quadrangle, $A + rB$ and $A + sB$ shall be the intersections of a transversal through B with one pair of opposite sides, A and $A + (r + s)B$ shall be the intersections of this transversal with the remaining pair of sides. It is easy to show that this construction is independent of the choice of the quadrangle, and that addition is commutative and associative.

(*c*) The relation between the numbers r, s, and rs. A transversal shall meet one pair of opposite sides of a complete quadrangle in A and B, a second pair in $A + rB$ and $A + sB$, while it meets a third pair in $A + B$ and $A + rsB$. This will make $rs = sr$, when, and only when, the construction yields a unique result, and this unicity of result is shown, in turn, to depend on Pappus's theorem. We thus reach in a new form a result, first exhibited by Hilbert, namely, that the

independence of Pappus's theorem from the other theorems of incidence may be shown by the existence of number systems where multiplication is not commutative, the calculus of quaternions being the classical example.

The second chapter of our book takes a fresh start, dealing with a much more restricted domain, namely, a limited region of real space. Peano's axiom which requires that a line that meets one side of a triangle, while it meets a second side produced, shall meet the third side, is very fundamental. This is really a most useful assumption for it accomplishes two distinct ends, it gives sufficient conditions for the intersection of coplanar lines, and for the projection of betweenness into betweenness. Betweenness is, in fact, a very fundamental relation in this sort of development, and we have at the proper moment a continuity axiom based, as usual, on a Dedekind cut among the points of a line-segment. With continuity we can do many things: prove Pappus's theorem, establish connection with the algebra of real numbers, etc. But our author does not remain content with his limited space but presently fills it out to the complete space of projective geometry, introducing ideal elements by a device first employed, we believe, by Pasch. Suppose, in fact, that in this limited space we have a system of lines of which at least one goes through every point, and of which each two are coplanar. An example of such a system is the assemblage of all lines through a point of the region. When the system is not so constituted, *i.e.*, when no two intersect in the region, then we say that they determine an ideal or postulated point which lies on all of them. Ideal lines and planes are then introduced in a similar fashion, and the whole of projective space is presently available. One rather wonders why Baker did things in just this way, why he did not follow in the real domain the plan already pursued in the general domain of assuming at the outset the intersection of coplanar lines, and postulating the laws of separation of four elements of a one-dimensional fundamental form. The explanation is, perhaps, to be found on p. 2: "In geometry, as applied to the external world, we cannot but be conscious for instance, in dealing with the points of a line, of the difference between the points which we regard as accessible, and those which we regard as the others." This would seem to mean that historically, and psychologically, geometry begins with real accessible points, and that it is natural to base the logical structure on these also. Yet the other method of approach is very neat.

The third chapter is in the nature of reconciliation of the two that preceded it. It is shown how the assumption of complex points, *i.e.*, points corresponding to complex numbers, rounds out and completes the real space of the second chapter. There is, here, a real obscurity arising from the writer's reluctance to state his assumption explicitly. We are told on p. 141 that Ch. III is a continuation of Ch. I. That suggests that the assumptions made in Ch. I and not those in Ch. II are valid. Then we read on p. 152: "Conversely, two spaces of a like number of dimensions, with a point-to-point correspondence of such a kind that to every incidence of elements in one space corresponds a precisely similar incidence of elements in the other, are necessarily related (projective) as we see by recalling the construction of related ranges."

This amounts to saying that every transformation that carries a point to a point and a plane to a plane, or a line to a line, is a projective transformation; true in the real domain, but untrue in the complex one. Witness the transformation which consists in interchanging conjugate imaginary elements.

The final section shows how complex geometry may be built out of real geometry by defining complex elements as groups of real ones. The first successful development of this kind was given by Von Staudt, his method being to define a complex point as an elliptic involution of real collinear points, together with a sense of description. Baker, however, follows a later method devised by Klein, where complex points are defined by cyclic projectivities among real ones. His treatment is based very directly on the exposition of Vahlen, and does not seem to us as clear as the original, but it is fair to say that he explicitly announces that all he says on this subject should be regarded as provisional.

What shall we say of the book as a whole? Is it as brilliant and stimulating as Hilbert's *Grundlagen der Geometrie?* No, it is not. Is it as searching and profound as Veblen and Young's *Projective Geometry?* No one could make such a claim. Is it as crisp and clear-cut as Vahlen's *Abstrakte Geometrie?* Such is certainly not the case. Does it afford as direct and natural an approach to the standard theorems as does Enriques' *Geometria Proiettiva?* Such is not our candid opinion. What then? We must judge a book finally by what the author sets out to do. We turn back to the preface to see what he expects of his book: "It seeks to set these relations (of points and lines and planes) in an ordered framework of deduction, gradually rendered comprehensive and precise enough to include all subsequent theory. To this end it puts aside at first most of the intricate details which make up the burden of what is generally called geometry. . . . Experience has shown that many students, especially of the class who look forward to becoming Engineers or Physicists, to whom the geometry of the usual text-books is tiresome, find such a course stimulating and easy, when the matter is properly presented to them. The mathematician who has followed such a course will find that he has no cause to think that the wrong things have been presented to him."

So the author wishes us to judge his book as an introduction, especially suited to those students who do not intend to specialize in mathematics. Our judgment must finally depend on the extent to which we agree with his educational theory. If he be correct in his opinion that the usual subject matter should be replaced by something more generalized, more abstract, capable of opening wider horizons, why then he deserves the great credit that should be awarded to every real pioneer in education. On the other hand, if his first assumption be essentially unsound, if truth be on the side of the older theory, which believed that the human mind proceeded naturally from the concrete to the abstract, and insisted that the natural approach to geometry is through the clean-cut, definite theorems of elementary metrics, why then we still welcome him as the first of his nation to write a scholarly text on a topic of recognized importance and value. May he speedily be followed by others, anxious to continue the good work.

JULIAN L. COOLIDGE.

NOTES.

In *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger, Juni 1919–Maj 1920*, Copenhagen, 1920, there is a sketch of H. G. ZEUTHEN (1839–1920) by A. C. Juel, pp. 67–77, accompanied by a splendid portrait-plate.

G. H. HARDY's remarkably interesting biography of Ramanujan, to which we have already referred more than once (*1921*, 458; *1922*, 19), is reprinted in *The Journal of the Indian Mathematical Society*, June, 1922.

In *The Cyclopædia of American Biography, new enlarged edition of Appleton's Cyclopædia of American Biography*, volume 9 (New York, The Press Association Compilers, Inc., 1922), there are biographies of EMORY MCCLINTOCK (see *1916*, 271; *1918*, 228), page 15, E. C. PICKERING (*1919*, 134), 329, and T. J. J. SEE, 363.

In *The Encyclopaedia and Dictionary of Education* edited by Foster Watson, 4 volumes (paged consecutively), London, Pitman, 1921–1922, the following mathematical topics, among others, are treated: Teaching of algebra, Alligation, Approximation, Arabic notation, Arithmetic, Averages, Axiom, Binomial theorem, Calculating machines by G. H. BRYAN, Casting out nines, Checks on computation, Cocker's arithmetic, Teaching of commercial arithmetic, Teaching of conic sections, Convergence, Decimal coinage, Finger manipulation, Non-Euclidean geometry, Teaching of geometry, Cartesian geometry, Graphs and graphic curves, The Grube method, Logical and historical schools of mathematics (modern) by P. E. B. JOURDAIN, Teaching of machine drawing, Significance of mathematical theory by G. H. BRYAN, History of mathematics by W. W. R. BALL, Metric system, Napier's rods (or Napier's bones), Teaching of projective geometry by J. L. S. HATTON, Drawing of sections, Tillich's bricks, and Trigonometry.

The following quotation from an article by Professor JAMES HENRY BREASTED of Chicago University is of interest to students of the history of mathematics. Professor Breasted refers indirectly to an article by Professor L. C. KARPINSKI, "Algebraical developments among the Egyptians and Babylonians", which appeared in this MONTHLY, *1917*, 257–265. The quotation is taken from an article on The Edwin Smith Papyrus which appears in the current number of the *Bulletin of the New York Historical Society*.

> "The current conclusion regarding the mind of the ancient Egyptian, a conclusion in which I have myself heretofore shared, has been that he was interested in scientific principles, if at all, solely because of the unavoidable necessity of applying them in practical life; that if he discussed the superficial content of a many-sided geometrical figure or the cubical content of a hemisphere it was because he was obliged to measure fields for taxation purposes and to compute the content of granaries. In the field of Egyptian mathematics Professor Karpinski of the University of Michigan has long insisted that the surviving mathematical papyri clearly demonstrate the Egyptians' scientific interest in pure mathematics for its own sake. I have now no doubt that Professor Karpinski is right, for the evidence of interest in pure science, as such, is perfectly conclusive in the Edwin Smith Medical Papyrus."

The first edition of the first volume of ARNE FISHER, *The Mathematical Theory of Probabilities and its Application to Frequency Curves and Statistical Methods*, appeared in 1915 (New York, Macmillan). The second edition of this volume (1922), enlarged by more than one hundred pages of new material, treats

of Mathematical probabilities, frequency curves, homograde and heterograde statistics. Mr. Fisher proposes to treat the methods of Pearson, Edgeworth, Kapteyn, Bachelier, and Knibbs, and to show their relation to Laplace's theory, in the second volume. Macmillan announced for publication in August, 1922, Arne Fisher, *An Elementary Treatise on Frequency Curves and their Application to the Construction of Mortality Tables* (225 + 15 pages).

We have already listed (*1920*, 218; *1921*, 269) fifteen of the volumes of the great edition of Euler's *Opera Omnia* now being issued under the auspices of the Swiss Society of Naturalists. Three more volumes have recently been published: (*a*) series I, volume 6: *Commentationes algebraicae ad theoriam aequationum pertinentes*, edited by F. RUDIO, A. KRAZER and P. STÄCKEL, 1921; (*b*) series I, volume 7: *Introductio in Analysin Infinitorum*, tome primus, edited by F. RUDIO, 1922; (*c*) series II, volume 14: *Neue Grundsätze der Artillerie . . . aus dem Englischen des . . . Benjamin Robins übersetzt und mit den nöthigen Erläuterungen und vielen Anmerkungen versehen von Leonhard Euler*, edited by F. R. SCHERRER, 1922.

Many readers will recall *The First Book of Geometry* (London, Dent, 1905; 16 + 222 pp.) written by GRACE CHISHOLM YOUNG, with the assistance of her husband W. H. YOUNG, for the use of her children. An attractive German edition by S. and F. Bernstein was issued by Teubner in 1908 (16 + 239 pp.). In 1921 a Hebrew translation by Dr. Elias Olschwanger was published by Wostock of Dresden (14 + 187 pp.; price, in Germany, bound, 67 marks).

This work is often referred to in connection with another publication appearing about the same time, namely C. A. LAISANT'S notable little book: *L'Initiation mathématique, ouvrage étranger á tout programme; dédié aux amis de l'enfance* (Paris, Hachette, 1906). Among various translations of it into other languages, an English edition appeared in 1913 (with the title *Mathematics*, London, Constable). Compare *1920*, 339, 498. W. H. Young is now professor of mathematics at University College, Aberystwyth, Wales.

The first half of the second edition of the Geometry volume of the admirable and most useful *Repertorium der höheren Mathematik*, edited by H. E. TIMERDING, was published by Teubner in 1910. This contained 24 chapters devoted to foundations and plane geometry. The second part on space geometry appeared in February, 1922 (chapters 25–42, pages 12 + 537–1165; price, in Germany, 96 marks, bound). O. STAUDE, of Rostock, is author of the first five chapters on surfaces of the second order and gauche curves of the third and fourth order. Six chapters are written by L. BERZOLARI, of Pavia, on the general theory and geometry of algebraic surfaces, surfaces of the third order, general theory of algebraic space curves, and special algebraic curves. TIMERDING, of Braunschweig, contributes the chapters on special surfaces of the fourth order, and on rational transformations of curves. K. ZINDLER, of Innsbruck, wrote the chapter on line geometry, and E. SALKOWSKI, of Hannover, contributed the chapters on space curves and developable surfaces, general theory of surfaces, and special classes of surfaces and systems of surfaces. The "Register" occupies pages 1134–1161, and "Berichtigungen und Zusätze," pages 1162–1165.

One of the most useful reference works for the mathematician is J. C. Poggendorff's *Biographisch-literarisches Handwörterbuch zur Geschichte der exacten Wissenschaften* of which volumes 1–2, covering the literature through 1857, appeared in 1863; the third volume, for the period 1858–1882, edited by B. W. Feddersen and A. J. von Oettingen was published in 1898; and the fourth volume, 1904, edited by Oettingen, 1883–1903. The manuscript of a fifth volume for 1904–1922 is now in course of preparation. Appeal has been made by the editors for information to be sent to them, at the earliest possible moment, addressed: Poggendorff-Büro, Beethovenstrasse 6 (Universitäts-Bibliothek), Leipzig, Germany. The information requested is as follows: surname and first names really used; "consanguineous relations of interest to science"; degrees and when and where obtained; address; year, month, day and place of birth; present and former positions with dates of appointment; "Emeritus? Retired? Since when?" Are you editor of a periodical? instruments invented or special achievements and year of publication or finishing; titles of books published by the author, either alone or in collaboration with other, stating year and place of publication of each work, its size as well as the total number of pages; titles of those articles and papers in the domain of the exact sciences which have appeared in periodicals or proceedings of Societies since 1903,—in each case indicating the periodical, the volume, the year of publication and total number of pages of the article.

ARTICLES IN CURRENT PERIODICALS.

JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER-VEREINIGUNG, volume 30, nos. 9–12, 1921: "Johannes Thomae" by H. Liebmann, 133–144; "Über paraboloidische Flächen" by P. Franck, 145–151; "Das Problem der 36 Offiziere" by H. F. MacNeish, 151–153; "Über stetige Funktionen mit überalldicht divergierender Fourierreihe" by L. Neder, 153–155; "Eine algebraische Behauptung von Gauss. II" by A. Loewy, 155–158; "Das Abelsche Gleichungsproblem bei Euler" by S. Breuer, 158–169; "Über die Hardy-Littlewoodschen Arbeiten zur additiven Zahlentheorie" by E. Landau, 179–185.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 14, February, 1922: "Focus-acnodal cubic" by P. Hemraj, 3–13; "On a particular type of similar triangles" by F. H. V. Gulasekharan, 14–18.

MATHEMATICAL GAZETTE, volume 11, March, 1922: "The structure of the atom" by J. W. Nicholson, 37–42; "Vectors" by C. Godfrey, 43–44—May: "Differential equations in mechanics and in physics" by A. R. Forsyth, 73–81 [Presidential address.]

MATHEMATICS TEACHER, volume 15, March, 1922: "The psychology of the equation" by E. L. Thorndike, 127–136; "Reaction vs. radicalism in the teaching of mathematics" by G. W. Myers, 137–146; "The definition of similarity" by G. W. Evans, 147–151; "The place of elementary calculus in senior high school mathematics" by N. B. Rosenberger, 152–156; "Experimental geometry" by G. A. Harper, 157–163; "How can I bring the soul of mathematics to my pupils?" by A. H. Huntington, 164–171; "Papers by pupils of the plane geometry classes of Fullerton Union High School" by Lena E. Reynold, 172–181; Discussion, 182–184; New books, 185–188; News and notes, 189–190—April: "Functionality in mathematical instruction in schools and colleges" by E. R. Hedrick, 191–207; "Calculus for schools" by W. H. Tyler, 208–211; "The psychology of problem solving" by E. L. Thorndike, 212–227 (to be concluded); "Cultural value of mathematics" by M. G. Kane, 228–236; Discussion, 237–240; News and notes, 241–245; "The Chicago meeting of the National Council of Teachers of Mathematics," 246–251; Research department, 251–252—May: "The psychology of problem solving" (conclusion) by E. L. Thorndike, 253–264; "The teaching of beginning geometry" by A. J. Schwartz, 265–282; "The origin of mathematics—a first lesson in secondary mathematics" by W. Betz, 283–293; "Robert Recorde" by F. Cajori, 294–302; "A list of reference books and magazines for teachers of mathematics" by W. D. Reeve, 303–307; Discussion, News and notes, etc., 307–316.

MATHEMATISCHE ZEITSCHRIFT, volume 12, nos. 3–4, 1922: "Some problems of 'Partitio Numerorum': IV. The singular series in Waring's problem and the value of the number $G(k)$" by G. H. Hardy and J. E. Littlewood, 161–188 (see *1921*, 219); "Über das Wachstum der Potenzreihen in ihrem Konvergenzkreise. I" by O. Töplitz, 189–200; "Zum Waringschen Problem" by E. Landau, 219–247—Volume 13, nos. 1–2, 1922: "Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen" by G. Szegö, 28–55; "Neue Lösungen der Einsteinschen Gravitationsgleichungen. A. Das Feld in der Umgebung eines langsam rotierenden kugelähnlichen Körpers von beliebiger Masse in 1. und 2. Annäherung" by R. Bach, 119–133; "Neue Lösungen der Einsteinschen Gravitationsgleichungen. B. Explizite Aufstellung statischer axialsymmetrischer Felder" by R. Bach (Mit einem Zusatz über das statische Zweikörperproblem by H. Weyl), 134–145; "Arithmetical equivalents for a remarkable identity between theta functions" by E. T. Bell, 146–152.

MATHESIS, volume 36, January, 1922: "Préface" by J. Neuberg and A. Mineur, 5–8 ["Il y a quarante et un ans, paraissait le premier numéro d'un modeste recueil mathématique à l'usage des écoles spéciales et des établissements d'instruction moyenne, dirigé par deux anciens élèves de l'École normale des sciences avec la collaboration de plusieurs professeurs belges et étrangers; recueil modeste par le nombre de pages, mais bientôt considérable par le nombre et la valeur des articles publiés. Son programme correspondait essentiellement à la partie élémentaire des mathématiques supérieures et à la partie supérieure des mathématiques élémentaires. . . . Le succès vint rapidement, dépassant les plus belles espérances des fondateurs; d'année en année, les collaborateurs furent plus nombreux; le programme de 1881 s'élargit de manière à comprendre tout l'enseignement mathématique des facultés des sciences et le nombre de pages que l'on avait promis de publier annuellement fut doublé. . . . dans la préface de 1911, l'histoire des trente premières années de *Mathesis* est relatée de cette manière: . . . Cinq ans après avoir écrit ces lignes, les épreuves infligées à la Belgique forcèrent les fondateurs de *Mathesis* à suspendre la publication du journal; . . . *Mathesis* restera fidèle à son passé; elle reprendra la tâche interrompue, contribuer au progrès de l'enseignement des mathématiques."]; "Paul Mansion (1844–1919)" by A. Demoulin, 9–12 [with portrait]; "Géométrie et mécanique" by J. Neuberg, 13–15; "Sur les contacts des sphères tangentes à quatre plans" by V. Thébault, 16–18; "Sur l'inversion et sur une surface cubique à quatre points doubles" by L. Godeaux, 19–23; "Archimède. À propos d'un ouvrage récent" by H. Bosmans, 24–27 [review of P. V. Eecke, *Les Œuvres complètes d'Archimède traduites du Grec en Français avec une Introduction et des Notes* (Paris and Bruxelles, 1921)]—February: "Sur les involutions cubiques *sibi* conjuguées" by C. Servais, 45–49; "Bibliographie du triangle et du tétraèdre" by J. Neuberg, 50–52 (to be continued); "Sur une généralisation d'un théorème de Droz Farny" by R. Goormaghtigh, 52–55; "Analysis situs. Une démonstration du théorème de Petersen" by A. Errera, 56–61—March: "Sur l'hypocycloïde à trois rebroussements" by P. De Lepiney, 77–81; "Relations remarquables entre cinq sphères" by R. Goormaghtigh, 81–86; "Sur les sphères podaires" by C. Servais, 87–89; "À propos d'une construction du point de Feuerbach" by A. M(ineux), 89–94—April: "Bibliographie du triangle et du tétraèdre" (continued) by J. Neuberg, 117–120; "Une surface réglée du huitième ordre" by M. Stuyvaert, 121–124; "Sur l'ellipse et le cercle de Nagel d'un triangle" by J. Neuberg, 125–128 (to be continued); "Formule relative aux combinaisons avec répétition" by M. Sterkens, 129–130—May: "Bibliographie du triangle et du tétraèdre" (continued) by J. Neuberg, 161–164; "Sur l'orthopole et sur la cubique des dix-sept points" by R. Goormaghtigh, 164–166; "Remarques sur l' 'Arithmétique' de Simon Stevin" by H. Bosmans, 167–174 [with facsimile of two pages]; "Sur l'ellipse et le cercle de Nagel d'un triangle" (continued) by J. Neuberg, 175–177; "Calcul d'un emprunt par obligations, avec lots" by M. Sapin, 178–179.

REVUE DE MATHÉMATIQUES SPÉCIALES, volume 32, April, 1922: "Note sur la transformation conforme" by M. Lapointe, 145–146.

REVUE SCIENTIFIQUE, volume 60, April 22, 1922: "Les professeurs de mathématiques du Collège de France: Humbert et Jordan; Roberval et Ramus" by H. Lebesgue, 249–262; "Débat sur la relativité" by M. Gandillot, 262–267 ["AVERTISSEMENT.—Dans ce débat sur la relativité, la théorie d'Einstein est defendue par quatre disciples, *Mathémate, Astronomian, Physician et Logomane*, dont les cultures sont différentes; elle est attaquée par *Quaerens* qui exprime ma propre opinion."].

SCHOOL SCIENCE AND MATHEMATICS, volume 22, May, 1922: "The uses of algebra in study and reading" by E. L. Thorndike and Ella Woodyard, 405–415 (to be continued); "Pro-

jective method in mathematics" by B. Cosby, 451–455; "Tangent lines among the Greeks" by F. Cajori, 463–464; "Trigonometry without similar triangles" by J. A. Nyberg, 467–470; "A new ellipsograph?" by J. A. van Groos, 471–472—June: "The uses of algebra in study and reading" by E. L. Thorndike and Ella Woodyard, 514–522; "Graphical trisection of an angle" by E. D. Pickering, 548–549.

SCIENCE, new series, volume 55, May 5, 1922: "The Edward C. Pickering memorial" by S. A. Mitchell, 467–469 [A fund for the Association of Variable Star Observers to be devoted to variable star research]—May 12: "Mathematical publications," 508–509; "The Eliakim Hastings Moore Fund" by A. Dresden, 510—May 26; "The Einstein equations of the solar field from the Newtonian point of view" by L. P. Eisenhart, 570–572.

AMERICAN DOCTORAL DISSERTATIONS.

J. D. BARTER, "The homogeneous vector function and determinants of the P-th class," *University of California Publications in Mathematics*, vol. 1, 1920, pp. 321–343. (University of California, 1917.)

TERESA COHEN, "Investigations on the plane quartic." Pp. 191–211. [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] (Johns Hopkins University, 1918.)

H. D. FRARY, "The Green's function for a plane contour." Pp. 11–25. [Reprinted from *American Journal of Mathematics*, vol. 42, 1920.] (University of Illinois, 1918.)

K. W. LAMSON, "A general implicit function theorem, with an application to problems of relative minima." Pp. 243–256. [Reprinted from *American Journal of Mathematics*, vol. 42, 1920.] (University of Chicago, 1917.)

FLORA E. LE STOURGEON, "Minima of functions of lines." Pp. 357–383. [Reprinted from *Transactions of the American Mathematical Society*, vol. 21, 1920.] (University of Chicago, 1917.)

WAYNE SENSENIG, "Concerning the invariant theory of involutions of conics." Pp. 111–122. [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] (University of Pennsylvania, 1919.)

G. W. SMITH, "Nilpotent algebras generated by two units, i and j, such that i^2 is not an independent unit." Pp. 143–164. [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] (University of Illinois, 1917.)

J. S. TAYLOR, "A set of five postulates for Boolean algebras in terms of the operation 'exception,'" *University of California Publications in Mathematics*, vol. 1, 1920, pp. 241–248. (University of California, 1918.)

L. E. WEAR, "On self-dual plane curves of the fourth order." Pp. 97–118. [Reprinted from *American Journal of Mathematics*, vol. 42, 1920.] (Johns Hopkins University, 1913.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2977. Proposed by FLORENCE P. LEWIS, Goucher College.

A point moves in such a way that its polars with respect to two given conics intersect at right angles. Prove that the locus of this intersection is a rational quartic curve through the circular points, and find its double points.

2978. Proposed by C. N. SCHMALL, New York City.

Given the equation of the general cubic $f(x, y) \equiv ax^3 + 3bx^2y + 3cxy^2 + dy^3 + a_1x^2 + 2b_1xy + c_1y^2 + a_2x + b_2y + c_2 = 0$, show that the three asymptotes of the curve will be concurrent if

$$\begin{vmatrix} a & b & a_1 \\ b & c & b_1 \\ c & d & c_1 \end{vmatrix} = 0.$$

2979. Proposed by V. M. SPUNAR, Chicago, Ill.

If a tangent be drawn from a variable point of an ellipse of length equal to n times the semi-conjugate diameter at the point, the locus of its extremity will be a concentric ellipse with semi-axes equal to $a\sqrt{n^2+1}$, $b\sqrt{n^2+1}$.

2980. Proposed by J. ROSENBAUM, Milford, Conn.

Locate a point such that the sum of its distances from the vertices of a given polygon shall be a minimum.

SOLUTIONS.

2876 [1921, 89]. Proposed by H. S. UHLER, Yale University.

Let a, d, n, and r denote respectively the refracting angle of a prism, the total deviation of the ray (in a principal plane) produced by the prism, the relative index of refraction of the material of the prism with respect to the single surrounding medium, and the angle of refraction at the first or incidence face of the prism.

(a) Deduce the (new) formula given below. (b) Use this formula to show (without the calculus) that the deviation has the least value when the ray is symmetrically situated with respect to the two refracting faces. (c) Show that, for minimum deviation, the formula reduces to the classical laboratory expression for determining the index of refraction.

$$\frac{\sin^2 \tfrac{1}{2}(a+d)}{\sin^2 \tfrac{1}{2}a} = n^2 + \frac{\sin \tfrac{1}{2}d \sin (a+\tfrac{1}{2}d) \sin^2 \tfrac{1}{2}(a+d) \sin^2 (r-\tfrac{1}{2}a)}{[\sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}(a+d) \cos^2 (r-\tfrac{1}{2}a) + \cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}(a+d) \sin^2 (r-\tfrac{1}{2}a)] \sin^2 \tfrac{1}{2}a}.$$

SOLUTION BY THE PROPOSER.[1]

Let i and i' denote, respectively, the angles which the segments of the ray outside the prism make with the normals at the points of incidence and emergence. Let r and r' be the corresponding angles inside the prism.

(a) From the geometry of the obvious diagram

$$r + r' = a$$

and

$$i + i' = a + d.$$

Put $r = \frac{1}{2}a + x$, and therefore $r' = \frac{1}{2}a - x$. Then

$$\sin i = n \sin (\tfrac{1}{2}a + x)$$

and

$$\sin i' = n \sin (\tfrac{1}{2}a - x).$$

Adding and subtracting, we have

$$2 \sin \tfrac{1}{2}(a+d) \cos \tfrac{1}{2}(i - i') = 2n \sin \tfrac{1}{2}a \cos x,$$
$$2 \cos \tfrac{1}{2}(a+d) \sin \tfrac{1}{2}(i - i') = 2n \cos \tfrac{1}{2}a \sin x;$$

whence

$$n^2 = \frac{\sin^2 \tfrac{1}{2}(a+d) \cos^2 \tfrac{1}{2}(a+d)}{\sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}(a+d) \cos^2 x + \cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}(a+d) \sin^2 x}$$

or

$$\frac{\sin^2 \tfrac{1}{2}(a+d)}{\sin^2 \tfrac{1}{2}a} = n^2 + \frac{[\cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}(a+d) - \sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}(a+d)] \sin^2 \tfrac{1}{2}(a+d) \sin^2 x}{[\sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}(a+d) \cos^2 x + \cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}(a+d) \sin^2 x] \sin^2 \tfrac{1}{2}a}$$
$$= n^2 + \frac{\sin \tfrac{1}{2}d \sin (a + \tfrac{1}{2}d) \sin^2 \tfrac{1}{2}(a+d) \sin^2 x}{[\sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}(a+d) \cos^2 x + \cos^2 \tfrac{1}{2}a \sin^2 \tfrac{1}{2}(a+d) \sin^2 x] \sin^2 \tfrac{1}{2}a},$$

[1] See *1921*, 1. The proof and the formulæ given in this solution apply directly to the more general case of the deviation of the projected ray; that is, to D' of the paper referred to.

which is the required formula.[1]

(*b*) Since a and d are positive acute angles, the last term cannot become negative and is zero only when $x = 0$. Therefore d will have a minimum value when $x = 0$, so that $r = r' = \frac{1}{2}a$, and $i = i'$, and the complete ray is symmetrically situated with respect to the prism.

(*c*) For this minimum value of d, say d_0, we have the classical laboratory formula

$$n = \frac{\sin \frac{1}{2}(a + d_0)}{\sin \frac{1}{2}a}.$$

2879 [1921, 89]. Proposed by E. J. OGLESBY, Washington Square College.

Given the values of $U_{5:9}$, $U_{5:10}$, $U_{5:11}$, $U_{6:9}$, $U_{6:10}$, $U_{6:11}$, $U_{7:9}$, $U_{7:10}$, $U_{7:11}$ where $U_{h:k} = \sqrt{hk}$, find the value of $U_{6.2:9.3}$ by interpolation.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

From the first three given terms we find by the method of differences[2] $U_{5:9.3} = 6.8190$, from the next three $U_{6:9.3} = 7.4698$ and from the last three $U_{7:9.3} = 8.0685$. Using these three values then as a series by the same method we find $U_{6.2:9.3} = 7.5937$.

Also solved by H. N. CARLETON.

2881 [1921, 89]. Proposed by E. B. ESCOTT, Oak Park, Ill.

If, in the polynomial $X^3 - 2$, we substitute $x^2 + x - 4$ for X, the given expression can be factored, that is, $X^3 - 2 \equiv (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6)$. Find a substitution for X so that the polynomial $X^3 + pX^2 + qX + r$ may be factored.

SOLUTION BY THE PROPOSER.

Let

$$X^3 + pX^2 + qX + r = (X - b)(X + c)^2 - a^2(X - d)^2. \tag{1}$$

Expanding the second member and equating coefficients of like powers of X, we get

$$a^2 + b - 2c + p = 0, \qquad 2a^2d - 2bc + c^2 - q = 0, \qquad \text{and} \qquad a^2d^2 + bc^2 + r = 0.$$

Eliminating b and d and solving for a^2 we get

$$a^2 = \frac{(3c^2 - 2pc + q)^2}{4(c^3 - pc^2 + qc - r)}.$$

Therefore, it is necessary and sufficient that[3]

$$c^3 - pc^2 + qc - r = n^2,$$

and we get by substitution

$$a = \frac{3c^2 - 2pc + q}{2n}, \qquad d = -\frac{c^3 - qc + 2r}{3c^2 - 2pc + q}, \qquad \text{and} \qquad b = -a^2 + 2c - p.$$

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[1] We might say

$$\frac{1}{n^2} = \frac{\sin^2 \frac{1}{2}a \cos^2 x}{\sin^2 \frac{1}{2}(a + d)} + \frac{\cos^2 \frac{1}{2}a \sin^2 x}{\cos^2 \frac{1}{2}(a + d)}$$

$$= \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} + \left[\frac{\cos^2 \frac{1}{2}a}{\cos^2 \frac{1}{2}(a + d)} - \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)}\right]\sin^2 x$$

$$= \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} + \frac{4 \sin \frac{1}{2}d \sin (a + \frac{1}{2}d) \sin^2 x}{\sin^2 (a + d)}$$

and use this formula for (*b*) and (*c*) instead of the formula in the text, d being a minimum when $[\sin^2 \frac{1}{2}a]/[\sin^2 \frac{1}{2}(a + d)]$ is a maximum.—EDITORS.

[2] See *1921*, 330.

[3] Thus it seems to be a necessary part of the hypothesis that there is a rational number $-c$ that will make the given polynomial equal to minus the square of a rational number.

Letting f denote the polynomial, if $f(-c) = -n^2$ we can write

$$f(X) = (X + c)^2\left[X - 2c + p + \left(\frac{f'(-c)}{2n}\right)^2\right] - \left[\frac{f'(-c)}{2n}(X + c) - n\right]^2,$$

which we can make the difference of two squares by putting $X - 2c + p + \left(\frac{f'(-c)}{2n}\right)^2 = (x + t)^2$.

—EDITORS.

Thus, a, b, c, and d are known. The right-hand member of (1) will be the difference of two squares, and may be factored if we put $X - b = (x + t)^2$; *i.e.*, $X = (x + t)^2 + b$.

In the example given, $p = q = 0$, $r = -2$. The value $c = -1$ gives $n = 1$; whence, $a = 3/2$, $d = 5/3$, $b = -17/4$. $X^3 - 2 = (X + \frac{17}{4})(X - 1)^2 - \frac{1}{4}(3X - 5)^2$. Letting $X + \frac{17}{4} = (x + \frac{1}{2})^2$; *i.e.*, $X = x^2 + x - 4$, and substituting, we have

$$X^3 - 2 = [(x + \tfrac{1}{2})(x^2 + x - 5) + \tfrac{1}{2}(3x^2 + 3x - 17)][(x + \tfrac{1}{2})(x^2 + x - 5) - \tfrac{1}{2}(3x^2 + 3x - 17)]$$
$$= (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6).$$

2885 [1921, 139].

If A, B, C, X, Y are given collinear points, construct Z so that $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, where $\{ABCX\}$ denotes the cross-ratio of the points A, B, C, X. [From *the Math. Tripos Exam.*, Cambridge, Eng., 1905.]

I. Solution by Gertrude I. McCain, Westminster College, New Wilmington, Pa.

If $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, then $\frac{AB}{BC} : \frac{AX}{XC} + \frac{AB}{BC} : \frac{AY}{YC} = \frac{AB}{BC} : \frac{AZ}{ZC}$. Dividing by $\frac{AB}{BC}$, $\frac{XC}{AX} + \frac{YC}{AY} = \frac{ZC}{AZ}$. But $XC = -CX = -(AX - AC)$, $YC = -CY = -(AY - AC)$, and $ZC = -CZ = -(AZ - AC)$. Making the substitutions and adding 2 to each side of the equation, we have $\frac{AC}{AX} + \frac{AC}{AY} = \frac{AC}{AZ} + 1$, or $\frac{1}{AX} + \frac{1}{AY} = \frac{1}{AZ} + \frac{1}{AC}$. Let $AX = \cot\theta$, $AY = \cot\varphi$, $AC = \cot\psi$, and $AZ = \cot\tau$. Considering AX, AY and AC as line functions, construct the angles θ, φ and ψ, using a circle with radius unity. Also construct the tangents of θ, φ, and ψ as line functions using the same circle.

It is now possible to lay off $\tan\theta$ and $\tan\varphi$ as lengths on a line and from their sum subtract $\tan\psi$.

But $\frac{1}{AX} + \frac{1}{AY} - \frac{1}{AC} = \frac{1}{AZ} = \tan\tau$. Cot τ or AZ may then be constructed as a line function, and the point Z located on the line with A, B, C, X, Y.

II. Solution by Otto Dunkel, Washington University.

After showing that $1/AX + 1/AY = 1/AZ + 1/AC$, it should be observed that this result shows that the two sets of points A, X, Y and A, Z, C have a common fourth harmonic point K. Hence K can be constructed from the first set and then Z as the harmonic conjugate of C with respect to A and K, by the quadrilateral construction, thus giving a non-metrical construction. This would then give a projective definition of the sum of two cross-ratios. This result may be obtained by use of abridged notation as follows:

If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\xi = 0$, $\eta = 0$ are the equations of the points A, B, C, X, Y, then the expressions to the left in these equations may be chosen so that

$$\beta = b\alpha + \gamma, \qquad \xi = x\alpha + \gamma, \qquad \eta = y\alpha + \gamma,$$

and then $\{ABCX\} = x/b$, $\{ABCY\} = y/b$. It follows that $(x - y)\alpha = \xi - \eta$, and if K is the harmonic conjugate of A with respect to X, Y, its equation may be written $(x - y)\kappa = \xi + \eta = 0$ or $(x - y)\kappa = (x + y)\alpha + 2\gamma$. Since $2\gamma = (x - y)\kappa - (x + y)\alpha$, Z, the harmonic conjugate of C with respect to K, A, may have its equation written $2\zeta = (x - y)\kappa + (x + y)\alpha = 0$ or $\zeta = (x + y)\alpha + \gamma$. Hence $\{ABCZ\} = (x + y)/b = \{ABCX\} + \{ABCY\}$.

Also solved by Nathan Altshiller-Court, J. W. Clawson, Arthur Pelletier, and F. L. Wilmer.

2887 [1921, 139]. Proposed by the late L. G. Weld.

A carpenter's square moves with its outer edges in contact with two round pegs of given equal diameters. Define the locus of the "heel" of the square.

Solution by J. B. Reynolds, Lehigh University.

Let the radius of either peg be r and the distance between their centers $\sqrt{2}a$. Let the axes be taken so that the centers of the circles are at the points $(a, 0)$ and $(0, a)$, and assume that the

square is placed above and to the right of the two circles, respectively. Also let θ be the angle which the side tangent to the former circle makes with the x-axis. The limiting positions of the square will be when the heel is just in contact with one of the circular pegs; that is, for example, when $a\cos\theta = r + a\sin\theta$, or $\theta = \frac{\pi}{4} - \sin^{-1}\frac{r}{\sqrt{2}a}$, so that the physical limits of θ are $\pm\left(\frac{\pi}{4} - \sin^{-1}\frac{r}{\sqrt{2}a}\right)$.

The points of tangency for any position of the square will be $(a - r\sin\theta,\ r\cos\theta)$ and $(r\cos\theta,\ a + r\sin\theta)$ and the equations of these tangents will be, respectively,

$$-(x-a)\sin\theta + y\cos\theta = r \qquad \text{and} \qquad x\cos\theta + (y-a)\sin\theta = r,$$

from which we get

$$x = -r(\sin\theta - \cos\theta) + a\sin\theta(\sin\theta + \cos\theta),$$
$$y = r(\sin\theta + \cos\theta) + a\sin\theta(\sin\theta - \cos\theta)$$

for the parametric equations defining the curve, with $-\left(\frac{\pi}{4} - \sin^{-1}\frac{r}{\sqrt{2}a}\right) \leqq \theta \leqq \left(\frac{\pi}{4} - \sin^{-1}\frac{r}{\sqrt{2}a}\right)$ for that portion subject to the physical limitations of the problem. By elimination of θ, we get for the general equation of the curve[1]

$$2r^2[x^2 + y^2 - 2a(x+y) + 2a^2] = [x^2 + y^2 - ax - ay]^2.$$

If in this $r = 0$, we have the circle $x^2 + y^2 = ax + ay$, as we should. For the above limits of θ to be correct, the shorter arm (18 inches) of the square must be $> \sqrt{(2a^2 - r^2)} - r$.

2895 [1921, 184]. Proposed by R. M. MATHEWS, Wesleyan University.

To construct an equilateral triangle with its vertices lying on: (*a*) any three coplanar lines; (*b*) three parallels in space; and (*c*) any three lines in space.

SOLUTION BY F. L. WILMER, Omaha, Neb.

(*a*) Let the three lines be l_1, l_2 and l_3. When they are not parallel we may suppose that the first two intersect in O. We bisect one of the angles between them and let the bisector meet l_3 in C. Then draw two lines from C, making angles of 30 degrees with CO (or with OC produced) and cutting l_1 and l_2 in A and B, respectively. ABC will be an equilateral triangle and there will generally be four triangles that can be constructed in this way.

When the three lines are parallel, draw a secant cutting them in P_1, P_2, P_3, respectively, and construct an equilateral triangle with P_1P_3 as one side and Q, say, as the third vertex. Rotate this triangle about P_2 until Q falls on l_2 and let the triangle in the new position be $P_1'Q'P_3'$. If we denote by A and B the points where $P_1'P_3'$ cuts l_1 and l_3, respectively, we shall find that the equilateral triangle ABC, constructed on AB on the same side of AB as Q', will satisfy the conditions of the problem; for $P_2P_1'/P_2A = P_2P_3'/P_2B$. Therefore P_2 is a center of similitude for the triangles $P_1'Q'P_3'$ and ABC, and C lies on P_3Q' or l_2.

(*b*) Consider the prismatic surface with edges l_1, l_2, l_3 and suppose that an equilateral triangle of side L can be placed with its three vertices on these three lines, respectively. Let a, b, c be the lengths of the sides of a right section $a \leqq b \leqq c$. Then

$$\sqrt{L^2 - a^2} = \sqrt{L^2 - b^2} + \sqrt{L^2 - c^2} \tag{1}$$

or

$$3L^4 - 2(a^2 + b^2 + c^2)L^2 + 16A^2 = 0, \tag{2}$$

where A is the area of the right section.

If L^2 is replaced by c^2, the expression in the first member of (2) reduces to $-(b^2 - a^2)^2$ and this shows that the equation in L^2 has two positive roots, one less than c^2 and the other greater than c^2 (unless $a = b$). The root greater than c^2 makes the three terms of (1) all real and satisfies (1) as written, since the first term of (1) is greater than or equal to each of the others.

(*c*) Partial solution—Instead of the three given lines, we may consider two non-coplanar lines and a point. Take the common perpendicular of the two for the z-axis and the bisectors of the angles made by their projections on the mid-plane as the axes of x and y. Then we may let

[1] If we take the line joining the centers of the circles as the x-axis, with the origin at the mid-point of that line, the equation will be $(x^2 + y^2 - b^2)^2 = 2r^2[x^2 + (y-b)^2]$, where $2b$ is the distance between the centers.—EDITORS.

(x_1, px_1, h), $(x_2, -px_2, -h)$ and (α, β, γ) be the vertices of an equilateral triangle of side L and there will be three expressions for L^2 which give the three equations

$$(1+p^2)(x_1^2+x_2^2) - 2(1-p^2)x_1x_2 = L^2 - 4h^2,$$
$$(1+p^2)x_1^2 - 2(\alpha+p\beta)x_1 = L^2 - \alpha^2 - \beta^2 - (\gamma-h)^2,$$
$$(1+p^2)x_2^2 - 2(\alpha-p\beta)x_2 = L^2 - \alpha^2 - \beta^2 - (\gamma+h)^2.$$

Eliminating x_1 and x_2 from these three equations, we have an equation of the fourth degree in L^2 with coefficients which are functions of α, β, γ.

NOTE BY THE EDITORS.—There are an infinite number of solutions in (a) when the lines are not parallel, as well as when they are parallel. This may be shown as follows: draw an equilateral triangle ABC with A on l_1 and B on l_2. With a given position of AB, the vertex C may lie on either side of AB. One at least of the straight lines OC will cut l_3 in the point C', unless these are both parallel to l_3 and so the same line, and this can be avoided by shifting the position of the triangle. The lines $C'A'$ and $C'B'$, drawn from C' to l_1 and l_2, parallel respectively to CA and CB, will be two sides of the triangle $A'B'C'$ which is equilateral and satisfies the conditions of the problem. The triangles $A'B'C'$ for different triangles ABC will vary in size, for if they were all equal the locus of C' would be an ellipse and not the line l_2.

In (c) the construction is not always possible. If, for example, the given point is at the origin, $\alpha = \beta = \gamma = 0$, then $x_2 = \pm x_1$, and $L^2 = 4h^2/(1-3p^2)$ or $4p^2h^2/(p^2-3)$. It is necessary in this case, therefore, that the acute angles between the projections of the two lines shall be less than 60 degrees.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, 69 Weymouth St., Providence, R. I.

Miss MAY J. SPERRY, of Knox College, has been appointed instructor at the University of Syracuse.

Miss MARGARET C. PACKER, of Brown University, has been appointed instructor at Hood College, Fredericksburg, Md.

Miss FRANCES M. MERRIAM, of Brown University, has been appointed instructor of mathematics at Wellesley College.

Assistant professor J. W. HARRELL, of Baylor University, Waco, Texas, is on leave of absence studying at Brown University.

MARCUS SKARSTEDT, of Augustana College, has been appointed professor of mathematics in Whittier College.

Miss MINNIE W. CALDWELL has been appointed head of the department of mathematics in Chowan College, Murfreesboro, N. C.

Associate Professor E. L. DODD will teach at Williams College during 1922–1923 on leave of absence from the University of Texas.

C. G. SIMPSON, of the Carnegie Institute of Technology, has been appointed professor of mathematics in the Milwaukee College of Electrical Engineering.

J. G. FOWLKES has been made assistant professor of education in the University of Wisconsin.

R. L. CHARLES, associate professor of physics at Lehigh University, has been appointed professor of physics and electricity at Franklin and Marshall College.

At Brown University, W. R. BURWELL, of the University of Tennessee, has been appointed assistant professor of mathematics and dean of freshmen; J. H. FITHIAN, of Taft School, Watertown, Conn., and H. C. HICKS, of the University of Chicago, have been appointed instructors.

Dr. F. C. TOUTON, lecturer in secondary education at the University of California, has been appointed associate professor of education at the University of Southern California. Dr. Touton is joint author of the Hawkes, Luby and Touton mathematical text-books.

Professor GLENN JAMES, of Carnegie Institute of Technology, has been appointed assistant professor of mathematics in the Southern Branch of the University of California, and Dr. F. C. LEONARD has been appointed instructor of mathematics and astronomy, in charge of the work in astronomy.

Professor W. J. RISLEY, of James Millikin University, has been appointed professor of mathematics in the Colorado School of Mines, and Mr. J. R. EVERETT, of Carnegie Institute of Technology, has been appointed assistant professor (see *1921*, 286).

F. A. WELLS, instructor of mathematics at the United States Naval Academy, has been appointed instructor of mathematics at the University of Virginia.

J. E. DAVIS, instructor of mathematics at the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Arkansas.

CLAIR REID, instructor of mathematics at Purdue University, has been appointed instructor at the University of Michigan.

T. B. HENRY, instructor of mathematics at the University of Kansas, has been appointed head of the department of mathematics in Highland College, Kansas.

Professor J. M. HOWIE, for twenty-four years head of the department of mathematics at the Nebraska State Teachers College, has been appointed professor of mathematics at Alma College.

Miss MARTHA P. MCGAVOCK, of Sullins College, has been appointed instructor of mathematics at Wellesley College.

Professor OSWALD VEBLEN, of Princeton University, has been granted leave of absence for the second semester of 1922–1923 and is to spend it in Europe.

Dr. C. E. STROMQUIST, of the University of Wyoming, is on leave of absence in California, necessitated by his ill health. Professor H. C. GOSSARD, who during the past two years has been engaged with the Y. M. C. A., has been appointed acting chairman of the department of mathematics.

In the mathematics department of the State College of Iowa, leave of absence during 1922–1923 has been granted to Associate professor JULIA T. COLPITTS to study at Cornell University, and Assistant professor E. C. KIEFER to study at the University of Michigan.

At the University of Colorado, Assistant professor G. H. LIGHT has been promoted to a full professorship of mathematics and Dr. CLARIBEL KENDALL to an assistant professorship.

At the University of Maine N. R. BRYAN has been appointed associate professor of mathematics for the year 1922–1923 in place of Professor L. S. HILL, who has been granted leave of absence and will study and act as instructor at Yale University.

Associate Professor EMMA L. KONANTZ, of Ohio Wesleyan University, has been

granted an extended leave of absence and in January will resume her teaching in Peking University and her study of early Chinese mathematics (see *1921*, 402; *1922*, 195).

Dr. J. D. BOND, of Louisiana State University, has been appointed associate professor of mathematics in the University of Tennessee. (See *1921*, 403.)

Professor W. E. PATTEN, of the Government Institute of Technology of Shanghai, a charter member of this Association, has been made professor of hydraulic engineering in Tangshan College, Tangshan, North China.

Dr. H. C. M. MORSE and Dr. W. L. G. WILLIAMS have been promoted to assistant professorships at Cornell University.

Dr. E. B. WILSON, professor of mathematical physics at the Massachusetts Institute of Technology, has been appointed professor of vital statistics at Harvard School of Public Health.

At the University of Kansas, Dr. U. G. MITCHELL was advanced to a full professorship for the year 1920–1921, and Dr. E. B. STOUFFER for the year 1921–1922.

At the University of Minnesota, Professor A. L. UNDERHILL has been granted a sabbatical leave of absence for the year 1922–1923. He expects to study in France. Mr. H. W. CHANDLER, of the University of Iowa, has been appointed instructor of mathematics.

At the Western College for Women, Oxford, Ohio, Miss Hazel E. SCHOONMAKER, of Gulf Park College in Mississippi, a graduate of Wellesley and an M. A. of Radcliffe, has been appointed head of the department of mathematics. Compare *1921*, 332.

At the University of Washington, Dr. E. T. BELL has been promoted to a full professorship, and Dr. L. L. SMAIL to an assistant professorship. Dr. HERMANCE MULLEMEISTER has been granted leave of absence for the year 1922–1923, which he will spend in Holland in study and research. Miss ECHO D. PEPPER, of the University of Washington, will take his place.

At the University of Michigan, Assistant Professor R. B. ROBBINS has returned for the year 1922–1923 after an absence of two years, during which period he has served as assistant actuary in the Insurance Departments of the State of Missouri and the State of New York. Mr. R. V. CHURCHILL, of the University of Chicago, has been appointed instructor of mathematics in the Engineering Department. Mr. C. C. CRAIG, of Indiana University, has been appointed instructor of mathematics in the College of Literature, Science and the Arts.

Professor C. H. RICHARDSON, of Georgetown University, Kentucky, taught in the summer session of the University of Michigan for 1922.

At Yale University, Assistant Professor LEIGH PAGE, of the department of physics, has been promoted to a full professorship of the mathematical sciences, with assignment to the Sheffield Scientific School and Dr. W. L. CRUM has been promoted to an assistant professorship of mathematics.

On account of ill health, Professor E. W. BROWN, of Yale University, had to give up his mathematical work about November 1. His advanced courses are to be conducted during the present year by Professor G. D. BIRKHOFF, of Harvard University.

FREDERICK ANDEREGG, professor of mathematics, emeritus, at Oberlin College since 1920 (compare *1920*, 336), died October 9, 1922. He was born in Meiringen, Switzerland, June 11, 1852, and was brought to America when ten years of age. Graduating A.B. from Oberlin in 1885, he was tutoring mathematics there 1885–1888. During 1888–1890 he was at Harvard (A.M. 1889) as a graduate student, and as instructor in mathematics during the latter year. Appointed associate professor of mathematics at Oberlin in 1890, he became professor in 1892. During 1903–1904 he studied at the University of Berne. Professor Anderegg's mathematical publications were not extensive. His pamphlet, *Algebra Problems introductory to College Mathematics* (Boston and New York, 1891), contained graded exercises for freshmen followed by examination papers from Harvard and Oxford, 1884–1886. A little article, "A perfect magic square," appeared in this MONTHLY, *1905*, 195–196. In collaboration with E. D. Roe, Jr., he was the author of *Trigonometry for Schools and Colleges* (Boston, 1896; revised edition, 1913). He was a charter member of the Association, had been a member of the American Mathematical Society since 1911, and was a member of the Swiss Alpine Club.

Mrs. SOPHIE (WILLOCK) BRYANT, aged 72, met her death by accident near Chamonix, Switzerland, some time between August 15 and August 28, 1922, when her body was found by Alpine hunters. She was born in Dublin and educated privately and at Bedford College, London, where she graduated with mathematical and moral science honors in 1881. She took the degree of doctor of science in moral sciences, being the first woman in the British Isles to take that degree. She was married at the age of 19 to Dr. William Hicks Bryant, of Plymouth, and after his death a year later renewed her work as a student and became mathematical mistress at the North London Collegiate School for Girls. From 1885 she was head mistress of this school until 1918, when she retired. She wrote many educational, religious, and scientific articles and articles on Ireland and the Irish language. She collaborated with Charles Smith in preparing an edition of Euclid's *Elements*, Books I to IV, VI, and XI (London, 1901) and a key to the same (London, 1902).

PIERRE JEAN BAPTISTE HENRI BROCARD, whose constant and valuable contributions have long been a prominent feature of *L'Intermédiaire des Mathématiciens*, and many other periodicals, died at his home in Bar-le-Duc, France, January 16, 1922. He was born only a short distance away, at Vignot, on May 13, 1845. A student of the Ecole Polytechnique, 1865–1867, he rose to the rank of lieutenant colonel of the territorial engineers (Génie territorial). In 1874 he was appointed in charge of the government meteorological service in Algiers. The list of his publications relative to meteorology, climatology, rural economics and hygiene, and natural sciences, 1860–1894, occupies over 60 printed pages. For the same period the printed list of his publications in physics and chemistry occupies 26 pages, and in pure and applied mathematics, and astronomy, 72 pages. These lists, preceded by a synopsis of his scientific activities, were issued, in 1895, in one volume at Bar-le-Duc, where he spent the last thirty years of his life. His very numerous contributions to the literature of mathematics were mainly

in connection with the geometry of the triangle (Brocard point and Brocard circle are well-known terms), history, and bibliography. In 1897–1899 he published and distributed at his own expense the very valuable *Notes de Bibliographie des Courbes Géométriques* (344 + 243 pages), now being elaborated into *Courbes Géométriques Remarquables (Courbes Spéciales) Planes & Gauches* by H. Brocard and T. Lemoyne, of which the first volume was published in 1919 (see *1920*, 130; *1921*, 218); the other two volumes are ready in manuscript but publication has so far been delayed by prohibitive cost of the same. Brocard's library was left to his friend André Gérardin, the editor and publisher of *Sphinx Œdipe*.

JACOBUS CORNELIUS KAPTEYN, born at Barneveld, Holland, January 19, 1851, died June 18, 1922. After taking his doctorate at the University of Utrecht in 1875 he became an observer at the observatory in Leyden. Since 1878 he had been professor of astronomy and mechanics at the University of Groningen. Among his very numerous publications, possibly those of chief interest to the mathematician are on "Skew frequency curves in biology and statistics" (Groningen, 1903 and 1916). Kapteyn visited Mount Wilson Observatory in 1910, and later, and a number of his astronomical papers are published in *Contributions from the Mount Wilson Observatory*, 1909–1920. Professor P. J. van Rhijh has been appointed his successor.

ERNEST RUDOLF KÖTTER, born in Berlin, Germany, August 7, 1859, died January 26, 1922. He took his doctorate at the University of Berlin where he was privatdozent for a number of years. From 1897 to the time of his death he was professor of descriptive geometry and graphical statics at the polytechnic school in Aix-la-Chapelle (Aachen). He is probably best known to mathematicians by his admirable report (28 + 486 pages), *Die Entwickelung der synthetischen Geometrie von Monge bis auf Staudt* (1847), which first appeared in *Jahresbericht der deutschen Mathematiker Vereinigung*, volume 5.

AXEL THUE, born in Tönsberg, Norway, February 19, 1863, died March 7, 1922. He received his education at the University of Christiania and was appointed a teacher in the technical school at Dortheim in 1894. In 1903 he was appointed professor of applied mathematics at the University of Christiania. His many published papers have been almost wholly in connection with problems of Diophantine analysis and other topics in the theory of numbers. Many of his results are listed in Dickson's *History of the Theory of Numbers*, volumes 1 and 2. In particular he found a general theorem applying to a particular problem discussed in this MONTHLY (see *1921*, 124; and *1922*, 159), namely the solutions of the equation $x^2 - y^3 = 17$.

Professor G. A. MILLER, of the University of Illinois, has been made an honorary member of the Indian Mathematical Society, together with Professor G. H. HARDY, of New College, Oxford, and Dr. G. T. WALKER, director-general of observations in India.

In April 1922 Professors W. F. OSGOOD and G. D. BIRKHOFF were elected corresponding members of the Royal Society of Sciences at Göttingen,

mathematical physical class. The list of corresponding members published in 1920 contains 80 names, including the following Americans: L. A. BAUER, Washington, D. C.; J. M. CLARKE, Albany, N. Y.; WILLIAM HILLEBRAND, Washington, D. C.; and R. W. WOOD, Baltimore, Md.

The gold medal of the Royal Astronomical Society has been awarded to J. H. JEANS, sometime professor of applied mathematics at Princeton University, for his contributions to the theory of cosmogony. Professor Jeans has already received one medal, awarded to him in 1919 by the Royal Society of London (see *1920*, 46).

E. T. WHITE, mathematics master of the London, Ontario, Normal School, was awarded the degree of doctor of pedagogy by the senate of the University of Toronto at its meeting on March 10.

The honorary degree of doctor of science was conferred upon Professor A. G. WEBSTER, of Clark University, at the commencement exercises of Princeton University.

A mathematical meeting and dinner in honor of Professor CHARLOTTE ANGAS SCOTT, on the completion of her thirty-seventh year as head of the department of mathematics in Bryn Mawr College, was held April 18, 1922, by Professor Scott's former students. The exercises consisted of an address of welcome by President M. CAREY THOMAS, an introductory address by Miss MARION REILLY, 1901, and a lecture by Professor A. N. WHITEHEAD, professor of applied mathematics in the Imperial College of Science, South Kensington, on "Relativity and gravitation, Group tensors and their application to the formulation of physical laws." After the lecture a tea was served at the deanery to about 200 guests.

At the dinner there were present former students, members of the American Mathematical Society, and members of the Bryn Mawr College faculty. Miss MARION REILLY acted as toastmistress. The speakers were: Professors E. H. MOORE, FORENCE BASCOM of Bryn Mawr, JAMES HARKNESS of McGill, E. W. BROWN, FRANK MORLEY, and A. N. WHITEHEAD. Professor Scott gave a response with an expression of appreciation. In regard to Professor Scott's service to Bryn Mawr College, Professor Bascom said in part, "It is this wisdom impartial, rational, creative, articulate, that Dr. Scott possesses in a marked degree. This is the quality which makes her judgment the one sought on all important faculty matters."

The honorary committee consisted of Professors R. C. ARCHIBALD, G. D. BIRKHOFF, E. W. BROWN, F. N. COLE, J. A. EIESLAND, JAMES HARKNESS, E. R. HEDRICK, FLORENCE P. LEWIS, Dean ISABEL MADDISON, Professors EMILIE N. MARTIN, HELEN A. MERRILL, E. H. MOORE, FRANK MORLEY, L. W. REID, R. G. D. RICHARDSON, E. J. TOWNSEND, OSWALD VEBLEN, H. S. WHITE, and RUTH G. WOOD.

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SEVENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

By invitation of the University of Rochester, the seventh summer meeting of the Association was held at this University on Wednesday and Thursday, September 6–7, 1922, in conjunction with, and immediately preceding, the summer meeting of The American Mathematical Society. There were 104 in attendance at the sessions, including the following 75 members of the Association:

N. H. Anning, University of Michigan.
R. C. Archibald, Brown University.
Florence L. Becker, High School, Webster, N. Y.
W. J. Berry, Brooklyn Polytechnic Institute.
William Betz, East High School, Rochester.
R. L. Borger, Ohio University.
W. G. Bullard, Syracuse University.
R. W. Burgess, Brown University.
W. D. Cairns, Oberlin College.
W. B. Carver, Cornell University.
E. H. Clarke, Hiram College.
H. E. Cobb, Lewis Institute.
A. B. Coble, University of Illinois.
C. E. Comstock, Bradley Polytechnic Institute.
Lennie P. Copeland, Wellesley College.
A. R. Crathorne, University of Illinois.
F. F. Decker, Syracuse University.
H. A. DoBell, Colgate University.
L. P. Eisenhart, Princeton University.
H. J. Ettlinger, University of Texas.
B. F. Finkel, Drury College.
C. A. Fischer, Trinity College.
W. B. Ford, University of Michigan.
R. E. Gaines, Richmond College.
A. S. Gale, University of Rochester.
R. E. Gilman, Brown University.
C. F. Gummer, Queen's University.
C. E. Harrington, University of Buffalo.
E. R. Hedrick, University of Missouri.
H. C. Hicks, Brown University.
T. H. Hildebrandt, University of Michigan.
W. A. Hurwitz, Cornell University.
Nelle L. Ingels, Interstate Commerce Commission.
L. C. Karpinski, University of Michigan.
O. D. Kellogg, Harvard University.
H. W. Kuhn, Ohio State University.
W. D. Lambert, U. S. Coast Survey.
Solomon Lefschetz, University of Kansas.
N. J. Lennes, University of Montana.
Anna D. Lewis, Lake Erie College.
T. R. Long, University of Rochester.
John Matheson, Queen's University.
G. A. Miller, University of Illinois.
Norman Miller, Queen's University.
G. R. Mirick, East High School, Rochester.
C. N. Moore, University of Cincinnati.
C. C. Morris, Ohio State University.
F. W. Owens, Cornell University.
Mrs. F. W. Owens, Cornell University.
L. R. Perkins, Middlebury College.
A. D. Pitcher, Western Reserve University.
L. C. Plant, Michigan Agricultural College.
V. E. Pound, University of Buffalo.
S. E. Rasor, Ohio State University.
F. W. Reed, Cornell University.
L. W. Reid, Haverford College.
Harris Rice, Worcester Polytechnic Institute.
R. G. D. Richardson, Brown University.
E. D. Roe, Syracuse University.
Hazel E. Schoonmaker, Western College.
W. H. Sherk, University of Buffalo.
W. G. Simon, Western Reserve University.
H. E. Slaught, University of Chicago.
Clara E. Smith, Wellesley College.
Virgil Snyder, Cornell University.
K. D. Swartzel, University of Pittsburgh.
W. A. Titsworth, Alfred College.
B. L. Waits, Florida A. and M. College.
C. W. Watkeys, University of Rochester.
J. H. Weaver, Ohio State University.
J. H. M. Wedderburn, Princeton University.
D. E. Whitford, University of Rochester.
F. B. Williams, Clark University.
C. H. Yeaton, Oberlin College.
J. W. Young, Dartmouth College.

From this list it will be noticed that a very good representation was afforded from the different regions covered by the Association. The joint dinner of the two organizations occurred on Thursday evening with Professor Eisenhart as presiding officer. Professor Archibald, the president of the Association, gave by